Statistics

Sample Space, Events & Probability

Experiment :- An operation or trial which can produce some well defined outcomes or output is known as experiment.

Example - Tossing a coin is an experiment and we have two possible outcomes — Head (H) or Tail (T)

Random Experiment :- A random experiment is an experiment or a process or a trial whose all the possible outcomes or results are known but the actual result that will occur can not be determined in advance.

Example - Rolling a die is an experiment and it has six possible outcomes (1, 2, 3, 4, 5, 6) but which face will actually turn up is not known.

Deterministic Experiment :- The experiments which have only one possible result or outcome i.e. whose result is certain or unique are called deterministic or predictable experiments.

Example - Newton's Laws of motion.

Non-Deterministic Experiments :- Experiments whose result is uncertain are called non-deterministic or unpredictable experperiments. There are more than one possible results or outcomes.

Example - Tossing a coin, rolling a die.

Sample-Space :- The set of all possible outcomes in a random experiment is called a sample space. It is denoted by S.

Example - In the throwing of a die, then the sample space

$S = \{1, 2, 3, 4, 5, 6\}$ and $n(S) = 6$

Discrete Sample Space :- If a sample space consisting of a finite number of possible outcomes which is countable then it is called discrete sample space.

A sample space, when an uncountable infinite set then it is known as continuous sample space

Events :- The outcomes or results (possible) of experiment (or trial) are called events. It is denoted by capital letters A, B, C ..... etc.

So, we can say that an event is a subset of sample space.

Example - If a coin is tossed, the outcomes head or tail are called events.

Types of Events :-

There are various types of events in probability. They are defined as—

1) **Simple or Elementory Event :-** An event consisting of single possible outcome is called a simple event.

Example - In a simultaneous toss of two coins, the event {HH} of getting both heads is a simple event.

2) **Compound Events :-** The joint occurrence of two or more simple events is called a compound event.

3) **Complementory Event :-** The complement of event A is denoted by A' or $\bar{A}$ or A^c

If an event A contains some points of a sample space, then the event complementary to A will contain the remaining points in that sample space.

Example - In tossing a coin, occurrence of Head and tail are complementary events.

The complement of an event A is event 'A does not occur' i.e. two complementary events are opposite to each other.

Favorable and Unfavorable Events :-

The number of outcomes of random experiment which results in the happening of a particular event are termed as the favorable event and not happening of event are unfavourable events.

i.e. Favorable events + unfavorable events = Total possible event

Mutually Exclusive Event :- Two or more events are said to be mutually exclusive or incompatible if the occurrence of one event does not affect the occurrence of other.

Two events A and B are said to be mutually exclusive or disjoint events if the intersection of A and B is a null set.

i.e. $\boxed{A \cap B = \phi}$

Example – Consider rolling of two dice and A be the event of getting the sum eleven whereas B be the event of getting the sum nine. So,

$A = \{(6,5), (5,6)\}$ and $B = \{(3,6), (4,5), (5,4), (6,3)\}$

Both are mutually exclusive.

Mutually Exhaustive Events :– A set of events is mutually exhaustive if atleast one of them must be occur.

Example – In rolling a die, the outcomes 1, 2, 3, 4, 5 and 6 are mutually exhaustive because any one of them must appear on the top.

Independent Events :– A set of events is said to be independ, if the occurence of any one of them does not affect the occurence of any other in the set

Otherwise the events are called dependent events.

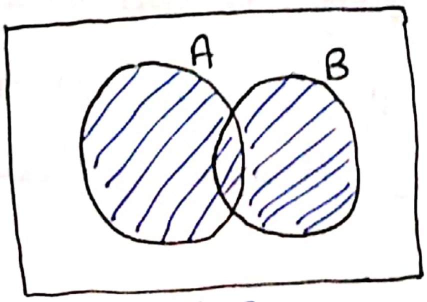
Example → 1) when a coin is tossed twice then the second toss will be not affected by the result of first toss. So, it is independent events.

2) If A draws a card from a full pack and does not replace it, then the draw afterward made by B is dependend on the draw made by A. Hence, they are dependend events.

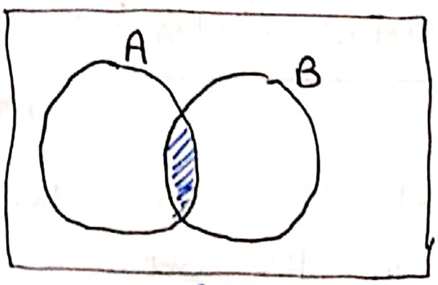
Union and intersection of two events :—

The union of two events A and B consists of all those outcomes that belongs to A or B or both A and B. It is denoted by $A \cup B$.

The intersection of two events A and B consists of those points which are common to A and B. It is denoted by $A \cap B$



$A \cup B$



$A \cap B$

Example - Let A be the event of getting an even number in the throw of a die and B be the event of getting a number less than 4,

then $A = \{2, 4, 6\}$ and $B = \{1, 2, 3\}$

So,

$A \cup B = \{1, 2, 3, 4, 6\}$

$A \cap B = \{2\}$

$A' = \{1, 3, 5\}$ and $B' = \{4, 5, 6\}$

Probability :- Probability is the study of radomness and uncertainty. Probability allows quantifying the chance that a certain event will occur.

The probability of an event is the number of favorable outcomes divided by the total number of outcomes possible.

General Rule for Probability :-

The general rule of the happening of an event is that if an event can happen in m ways and fail to happen in n ways, then the probability (p) of the happening of the event (A) is given by -

$$P(A) = \frac{m}{m+n}$$

i.e. Number of cases favourable to the event A divided by the total number of ways in which the event A can happen.

The probability of non-occurrence of the event A

$$P(A') = \frac{n}{m+n}$$

Here,

$$P(A) + P(A') = \frac{m}{m+n} + \frac{n}{m+n}$$

$$= \frac{m+n}{m+n} = 1$$

or

$$\boxed{P(A) + P(A') = 1}$$

Example - Two unbiased dice are thrown. Find the probability that -

(i) getting a sum of 6.

(ii) the numbers shown are equal.

(iii) the difference of numbers shown is 1.

(iv) the first die shows 6.

(v) the total of numbers greater than 8.

Sol - The two dice can be thrown in 6×6 ways
$= 36$ ways

So, $n(S) = 36$ where S is the sample space.

(i) Let E_1 be the events of getting a sum of 6.

So, $E_1 = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$

$\therefore \quad n(E_1) = 5$

Hence, the required probability $= \dfrac{n(E_1)}{n(S)}$

$= \dfrac{5}{36} = 0.1388$

(ii) Let E_2 be the events of getting equal number.

$E_2 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$\therefore \quad n(E_2) = 6$

Hence, the required probability $= \dfrac{n(E_2)}{n(S)}$

$= \dfrac{6}{36} = \dfrac{1}{6} = 0.16667$

(iii) Let E_3 be the events of showing the difference of numbers is 1.

So, $E_3 = \{(1,2), (2,1), (3,2), (2,3), (3,4), (4,3), (4,5), (5,4), (5,6), (6,5)\}$

and $n(E_3) = 10$

Therefore, the required probability $= \dfrac{n(E_3)}{n(S)}$

$$= \frac{10}{36} = 0.2777$$

(iv) Let E_4 be the event of first die showing 6.

So, $E_4 = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$$\therefore \quad n(E_4) = 6$$

Hence, the required probability $= \dfrac{n(E_4)}{n(S)}$

$$= \frac{6}{36} = 0.16667$$

(v) Let E_5 be the event of total numbers greater than 8.

So, $E_5 = \{(3,6), (6,3), (4,5), (5,4), (5,5), (4,6), (6,4), (5,6), (6,5), (6,6)\}$ i.e. $n(E_5) = 10$

So, the required probability $= \dfrac{n(E_5)}{n(S)}$

$$= \frac{10}{36} = 0.2777$$

Example – A bag contains 6 black and 8 white balls. One ball is drawn at random. What is the probability that the ball drawn is white?

Sol –

Total number of balls $= 6 + 8$

$= 14$

Number of white balls $= 8$

So, the required probability $= \frac{8}{14}$

$= 0.5714$ **Ans**

Classical or a Priori Probability or Mathematical Definition of Probability –

The classical approach to probability expresses probability as the ratio of the number of favorable outcomes in a series of successive trials to the number of total possible outcomes.

The probability of an event E to occur is the ratio of the number of cases in its favour to the total number of cases.

i.e. $P(E) = \dfrac{\text{number of cases favourable to event } E}{\text{Total number of cases}} = \dfrac{n(E)}{n(S)}$

Let an event can occur in 'a ways' and fail to occur in 'b ways' then probability of the event occuring is

$$P(E) = \frac{a}{a+b}$$

and probability of the event not occuring is

$$P(E') = \frac{b}{a+b}$$

So,

$$P(E) + P(E') = 1$$

Hence,

probability of occurence + probability of non-occurence = 1

and

$$0 \leq P(E) \leq 1$$

__Addition Theorem of probability__ :- Addition theorem is used to find the probability of happening of atleast one of the given events. It is based on following two conditions or events—

1) Mutually Exclusive Events
2) Non-mutually exclusive Events

1) **Mutually Exclusive Events** :- Two or more events are said to be mutually exclusive events if they don't have any element in common. i.e. intersection of two mutually exclusive events is null set.

Let A and B are two mutually exclusive events

then $P(A \cap B) = \phi = 0$ —— (1)

we know that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or $\boxed{P(A \cup B) = P(A) + P(B)}$ (from eq (i))

If A, B and C are three mutually exclusive events

then $\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$

Example - The probability of a horse A winning a race is 1/5 and the probability of a horse B winning the same is 1/4. what is the probability that -

(i) either of them will win

(ii) none of them will win.

Sol Let E_1 and E_2 be the events of winning the race by horse A and B respectively.

given that $P(E_1) = 1/5$ and $P(E_2) = 1/4$

(i) Since both of them can not win together.

So, E_1 and E_2 are mutually exclusive events.

$$\therefore \quad P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

$$= \frac{1}{5} + \frac{1}{4}$$

$$= \frac{9}{20} = 0.45$$

(ii) Probability of none of the horse winning is

$P(E_1 \cup E_2)'$ = 1 − probability of either of them will win

$$= 1 - P(E_1 \cup E_2)$$

$$= 1 - \frac{9}{20} = \frac{11}{20} = 0.55$$

2) Non-mutually Exclusive Events :- Two or more events are said to be non-mutually exclusive if the occurence of

one event affect the occurence of another events.

i.e. $P(A \cap B) \neq \phi$

If A and B are two events then the probability that at least one of them occur is denoted by $P(A \cup B)$ and defined as –

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

For three events A, B and C

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

Example– The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will not get an electric contract is 5/9. If the probability of getting atleast one contract is 4/5 then what is the probability that he will get both the contracts?

Sol – Let A = event that the contractor will get the plumbing contract

and B = he will get an electric contract.

Given that

$$P(A) = \frac{2}{3}, \quad P(B') = \frac{5}{9} \ \& \ P(A \cup B) = \frac{4}{5}$$

we have $\quad P(B) = 1 - P(B')$

$$= 1 - \frac{5}{9} = \frac{4}{9}$$

So,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or $\quad \frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P(A \cap B)$

or $\quad P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5}$

$$= \frac{30 + 20 - 36}{45} = \frac{14}{45} = 0.3111 \text{ Ans}$$

Multiplication Theorem of Probability :- Multiplication theorem of probability is used to find the probability, that all the events are occur. It is also based on two conditions -

1) **Multiplication theorem of probability for Independent Events -**

— The theorem states that the probability of the simultaneous occurence of two events that are independent is given by the product of their individual probabilit

Let A and B are two independent events, then the probability that both of them occur is

$$\boxed{P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)}$$

Similarly for three independent events A, B and C is

$$\boxed{P(A \text{ and } B \text{ and } C) = P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)}$$

Note – If A and B are independent events then the complements of A and B are also independent.

i.e. $P(A \cap B)' = P(A') \cdot P(B')$

Example – From a pack of 52 cards, two cards are drawn at random one after the other with replacement. What is the probability that both cards are kings?

Sol – Total number of cards = 52

from a pack of 52 card, there are 4 king

So, the probability of drawing a king $P(A) = \frac{4}{52}$

and the probability of drawing again the king after replacement $P(B) = \frac{4}{52}$

since, the two events are independent, then the probability of drawing two kings is —

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$$

$$= \frac{4}{52} \cdot \frac{4}{52}$$

$$= \frac{1}{169} = 0.0059 \quad \underline{\text{Ans}}$$

<u>Example</u> - A bag contains 4 red balls, 3 white balls and 5 black balls. Two balls are drawn one after the other with replacement. Find the probability that first is red and the second is black.

<u>Sol</u>

Total number of balls $= 4+3+5$

$= 12$

Probability of red ball in the first draw $P(A) = \frac{4}{12}$

and probability of a black ball in the second draw $P(B) = \frac{5}{12}$

since, both are independent events.

Hence, the probability that first is red and second are black will be

$$P(A \cap B) = P(A) \cdot P(B)$$

$$= \frac{4}{12} \cdot \frac{5}{12}$$

$$= \frac{5}{36} = 0.1388$$ Ans

2) Multiplication Theorem of probability for dependent Events –

If A and B are two dependent events, i.e. the occurrence of one event is affected by the occurrence of the other event then the probability that both A and B will occur is –

$$\boxed{P(A \cap B) = P(A) \cdot P(B/A)}$$

where $P(B/A)$ is the probability of B when A is already occurred.

For three dependent events A, B and C

$$\boxed{P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A,B)}$$

Example – A bag contains 5 white and 3 red balls and four balls are successively drawn and are not replaced. what is the chance that –

(i) white and red balls appear alternatively. and

(ii) red and white balls appear alternatively.

Sol – Total number of balls $= 5+3 = 8$

(i) The probability of drawing white balls first $P(A) = \frac{5}{8}$

the probability of drawing red balls $P\left(\frac{B}{A}\right) = \frac{3}{7}$

again probability of drawing a white ball $P\left(\frac{C}{A,B}\right) = \frac{4}{6}$

& probability of drawing a red balls again $P\left(\frac{D}{A,B,C}\right) = \frac{2}{5}$

since events are dependend then

$$P(A \cap B \cap C \cap D) = P(A) \cdot P\left(B/A\right) \cdot P\left(C/A,B\right) P\left(\frac{D}{ABC}\right)$$

$$= \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5}$$

$$= \frac{1}{14} \doteq 0.0714$$

Example - A bag contains 5 white and 4 black balls. A ball is drawn from this bag and are not replaced. what is the probability that two balls are different colors.

Sol - Total no of balls = 5+4 = 9

There are two possibility -

(i) First ball is white and the second ball is black and

(ii) first ball is black and the second ball is white

and events are dependent, because balls are not replaced

So, (i) probability of drawing first ball white and the second ball is ~~white~~ black $= \frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

(ii) probability of drawing first ball black and the second ball is white $= \frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

Since, these probability are mutually exclusive then using addition theorem

Probability that two balls are of different colors

$$= \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = 0.5555$$

Conditional Probability :- The probability of an event B occuring when it is known that some event A has already occured is known as conditional probability and denoted by P(B/A) and defined by -

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}} \quad , \quad P(A) > 0$$

Similarly, P(A/B) defines the conditional probability of A when B has already occured and

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \quad , \quad P(B) > 0$$

Example - Two dice are thrown. Find the probability that sum of the numbers coming up on them is 9, if it is known that the number 5 always occurs on the first dice.

Sol Let S be the sample space.

So, $n(S) = 6 \times 6 = 36$

Let E_1 = the event that sum of the number coming up is 9

& E_2 = the event occurence of 5 on the first dice

$\therefore \quad E_1 = \{(3,6), (6,3), (5,4), (4,5)\}$

$n(E_1) = 4$.

and $\quad E_2 = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

i.e. $\quad n(E_2) = 6$

So, $\quad E_1 \cap E_2 = \{(5,4)\} \quad \therefore \quad n(E_1 \cap E_2) = 1$

Now,

$$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{1}{36}$$

and

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Therefore,

$$P(E_1 / E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$= \frac{1/36}{1/6} = \frac{1}{6} \quad \underline{\text{Ans}}$$

Bayes' Theorem :— Bayes' theorem is introduced by Thomas Bayes an English Mathematician in his work known as Bayesian Decision Theory.

The Bayes theorem describes the probability of an event based on the prior knowledge of the conditions that might be related to the event.

Since we known the conditional probability $P(A/B)$, we can use the bayes rule to find the reverse probabilites $P(B/A)$

Let $A_1, A_2, \ldots A_n$ are n mutually exclusive and exhaustive events in the sample space S and B is another in S then

$$P(A_K/B) = \frac{P(A_K) \cdot P(B/A_K)}{P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) + \cdots + P(A_n) \cdot P(B/A_n)}$$

or

$$P(A_K/B) = \frac{P(A_K) \cdot P(B/A_K)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

For two mutually exclusive and exhaustive events A_1 and A_2 is

$$P(A_1/B) = \frac{P(A_1) \cdot P(B/A_1)}{P(A_1)\, P(B/A_1) + P(A_2) \cdot P(B/A_2)}$$

and

$$P(A_2/B) = \frac{P(A_2) \cdot P(B/A_2)}{P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2)}$$

Example – A bag P contains 2 white and 3 red balls and a bag Q contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from the bag Q.

Sol – Let A_1 = the event of ball being drawn from bag P

A_2 = " " " " " " " " Q

and B = The event of ball being red

Since, A_1 and A_2 are equally likely then

$$P(A_1) = P(A_2) = \frac{1}{2}$$

and $P(B/A_1) = \frac{3}{5}$ & $P(B/A_2) = \frac{5}{9}$

$\therefore$ probability of drawn the ball from the bag Q which is red

$$P\left(\frac{A_2}{B}\right) = \frac{P(A_2)\cdot P(B/A_2)}{P(A_1)\cdot P\left(\frac{B}{A_1}\right) + P(A_2)\cdot P\left(\frac{B}{A_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{\frac{5}{9}}{\frac{3}{5}+\frac{5}{9}}$$

$$= \frac{25}{52}$$ Ans