Permutations – Permutations is the arrangements of given number of elements taken one by one, or some, or all at a time. (order matter)

Let n and r are positive integer such that $1 \le r \le n$ then the number of permutations of n different things taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$ and defined as

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

Example - How many different numbers can be formed out of the digits 1, 2, 3, 4, 5, 6 taking four of them at a time.

Sol Here, No of digit $= n = 6$
and $r = 4$

So, Number of arrangements $= {}^nP_r$

$$= {}^6P_4$$

$$= \frac{6!}{(6-4)!}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!}$$

$$= 6 \times 5 \times 4 \times 3 = 360 \quad \underline{\text{Ans}}$$

Rule-1

Theorem - The number of all permutations of n distinct things taken all at a time is n! (without repetition)

Proof - Since, we know that number of permutations of 'n' different things taken 'r' at a time is

$$^nP_r = \frac{n!}{(n-r)!}$$

Since, we take all at a time then $r = n$

So, from eq (i)

$$^nP_n = \frac{n!}{(n-n)!}$$

or

$$^nP_n = \frac{n!}{0!}$$

or

$$\boxed{^nP_n = n!} \quad (\text{since } 0! = 1)$$

Example - How many words can be formed by using the word 'DELHI' ?

Sol - The word 'DELHI' has 5 different alphabets.

So, number of words taking all at a time

$$= n!$$

$$= 5!$$

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120 \quad \underline{\text{Ans}}$$

Rule-2 - The (with repetition) number of permutations of n things, taken all at a time, of which p are like one type, q alike of second type and so on (that means out of n things, Some are same type) then

$$\text{Total Number of permutations} = \frac{n!}{p!\,q!}$$

Example - How many words can be formed by using letters of the word 'MATHEMATICS'?

Sol

No of letters $n = 11$

in which 2M, 2A, 2T are present.

$$\text{Total no of words} = \frac{11!}{2!\,2!\,2!}$$

$$= \frac{11 \times 10 \times 9 \times \cancel{8} \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 2}$$

$$= 110 \times 63 \times 30 \times 24$$

$$= 6930 \times 720$$

$$= 4989600$$

Rule-3 The no of permutations of n different things, taken r at a time, when each may be repeated any no of times in each arrangement is n^r.

Rule-4 circular permutations –

The number of circular permutations of n distinct object is $(n-1)!$

Combinations :- The word combination means selection.

A selection of any r things out of n things is called a combination of n things r at a time. It is denoted by ^nC_r or $C(n, r)$ or $\binom{n}{r}$ and defined as –

$$\boxed{^nC_r = \frac{n!}{r!\,(n-r)!}}$$

(order does not matter)

Example- In how many ways, can we select a team of 4 students from a given choice of 15?

Sol - Here, $n = 15$

$r = 4$

So, number of possible ways of selection

$$= {}^{15}C_4$$

$$= \frac{15!}{4!\,(15-4)!}$$

$$= \frac{15!}{4!\ 11!}$$

$$= \frac{15 \times 14 \times 13 \times 12 \times 11!}{4 \times 3 \times 2 \times 1 \times 11!} = 1365 \text{ Ans}$$

Theorem - Prove that -

$$ {}^{n}C_{r} \cdot r! = {}^{n}P_{r} $$

Proof - we know that,

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} \quad \text{——— (I)} $$

and $$ {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!} \quad \text{——— (II)} $$

from eq (i) and (ii), we get

$$^nC_r = \frac{1}{r!}\,{}^nP_r$$

$$\Rightarrow \quad \boxed{^nC_r \cdot r! = {}^nP_r}$$

proved

Theorem - Prove that-

$$^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

Proof - L.H.S. $\quad {}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{r\,(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-\cancel{r}+1+\cancel{r}}{r\,(n-r+1)}\right]$$

$$= \frac{(n+1)\,n!}{r\,(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{(n+1)!}{r!\,(n+1-r)!} = {}^{n+1}C_r$$ R.H.S. proved

Note – if $^nC_x = {^nC_y}$

then either $x = y$ or $x + y = n$

Example – If $^{48}C_{12} + {^{48}C_{13}} + {^{49}C_{14}} = {^{50}C_x}$ then find x.

Sol – $^{48}C_{12} + {^{48}C_{13}} + {^{49}C_{14}} = {^{50}C_x}$

$\Rightarrow \quad {^{49}C_{13}} + {^{49}C_{14}} = {^{50}C_x} \qquad (\because {^nC_r} + {^nC_{r-1}} = {^{n+1}C_r})$

$\Rightarrow \quad {^{50}C_{14}} = {^{50}C_x} \qquad$ (again $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$)

$x = 14$ or $x + 14 = 50$ (since $^nC_x = {^nC_y}$ then $x = y$ or $x + y = n$)

$\Rightarrow \quad \boxed{x = 14 \text{ or } x = 36}$ Ans

Example – If $^nP_r = 720$ and $^nC_r = 120$ then find r.

Sol – we know that –

$^nC_r \times r! = {^nP_r}$

$\Rightarrow \quad 120 \times r! = 720$

$\Rightarrow \quad r! = 6 = 3!$

So, $r = 3$ Ans

Example - $^{n}P_{13} : {}^{n+1}P_{12} = \frac{3}{4}$ then find n.

Sol - we know that $^{n}P_{r} = \frac{n!}{(n-r)!}$

Now,

$$^{n}P_{13} : {}^{n+1}P_{12} = \frac{3}{4}$$

$$\frac{n!}{(n-13)!} : \frac{(n+1)!}{(n+1-12)!} = \frac{3}{4}$$

$$\frac{n!}{(n-13)!} \times \frac{(n-11)!}{(n+1)!} = \frac{3}{4}$$

$$\frac{n!}{(n-13)!} \times \frac{(n-11)(n-12)(n-13)!}{(n+1)\ n!} = \frac{3}{4}$$

$$\frac{(n-11)(n-12)}{n+1} = \frac{3}{4}$$

$$\Rightarrow 4(n-11)(n-12) = 3(n+1)$$

$$\Rightarrow 4(n^2-23n+132) = 3n+3$$

$$\Rightarrow 4n^2-95n+525 = 0$$

$$\Rightarrow 4n^2-60n-35n+525 = 0$$

$$\Rightarrow 4n(n-15)-35(n-15) = 0$$

$$(n-15)(4n-35) = 0$$

$$n = 15, \quad n = \frac{35}{4} \text{ (not valid)}$$

So, $n = 15$

Example – Find out r, if $^{12}P_r = 1320$

Sol – since, $^nP_r = \frac{n!}{(n-r)!}$

we have $^{12}P_r = 1320$

$$\frac{12!}{(12-r)!} = 12 \times 11 \times 10$$

or $$\frac{12!}{(12-r)!} = \frac{12 \times 11 \times 10 \times 9!}{9!}$$

or $$\frac{12!}{(12-r)!} = \frac{12!}{9!}$$

or $$(12-r)! = 9!$$

or $$12 - r = 9$$

$$r = 12 - 9 = 3$$ Ans

Example – In how many ways can a group of 5 members be formed by selecting 3 boys out of 6 and 2 girls out of 5?

Sol - Number of ways 3 boys can be selected out of 6 = $^6C_3$

$$= \frac{6!}{3!\,3!}$$

$$= \frac{6 \times 5 \times 4 \times 3!}{3! \times 3 \times 2} = 20$$

Number of ways 2 girls can be selected out of 5

$$= {^5C_2}$$

$$= \frac{5!}{2!\,3!}$$

$$= \frac{5 \times 4 \times 3!}{2! \times 3!} = 10$$

Therefore, total number of ways of selection

$$= 20 \times 10 = 200 \text{ Ans}$$

Example - How many different words can be formed with the letters of word 'EQUATION' so that -

1) The words begin with E ?
2) The words end with T ?
3) The words begin with E and end with T ?
4) The words begin with a vowel ?

Sol- The word 'EQUATION' contains 8 different words where no. of vowels are 5 and 3 consonants.

1) we fix E at start the first place.

So, remaining letters = 7

Therefore, No. of different words which begins with E

$$= {}^7P_7$$

$$= 7!$$

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 \text{ ways}$$ Ans

2) Now, we fix T at the end place.

$\therefore$ remaining letters = 7

So, Remaining 7 letters can be arranged by (end with T)

$$= {}^7P_7$$

$$= 7!$$

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 \text{ ways}$$ Ans

3) Here, we fix E at first place and T at the last place.

So, remaining 6 letters can be arranged by

$= {}^6P_6$

$= 6!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways Ans

4) Total no of vowels in 'EQUATION' is 5

So, first place can be occupied by any one of these vowels by 5 ways

and remaining 7 places can be arranged by

$= {}^7P_7$

$= 7!$

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Hence, total number of words begin with a vowels

$= 5 \times 5040$

$= 25200$ Ans

Example - From a group of 7 men and 6 women, five persons are to be selected to form a committee so that atleast 3 men are there on the committee then how many ways can it be done?

Sol – Given that – 7 men and 6 women

Prob – 5 persons are to be selected to form committee so that 3 men are there

So, we select such that
(3 men and 2 women) or (4 men and 1 woman) or (5 men only)

$$= {}^7C_3 \times {}^6C_2 + {}^7C_4 \times {}^6C_1 + {}^7C_2$$

$$= \frac{7!}{3!\,4!} \times \frac{6!}{2!\,4!} + \frac{7!}{4!\,3!} \times \frac{6!}{1!\,5!} + \frac{7!}{2!\,5!}$$

$$= \frac{7\times6\times5}{3\times2} \times \frac{6\times5}{2\times1} + \frac{7\times6\times5}{3\times2} \times \frac{6}{1} + \frac{7\times6}{2\times1}$$

$$= 525 + 210 + 21 = 756 \text{ ways}$$ Ans

Example – Out of 4 officers and 8 clerks in an organisation, a commitee of 5 consisting of 2 officers and 3 clerks is to be formed. In how many ways can this be done if –

1) There is no restriction on the selection.
2) One particular clerk must be on the committee ?

Sol - Given that - 4 officers and 8 clerks

1) The number of ways of selecting 2 officers and 3 clerks from a group =

$$^4C_2 \times {}^8C_3$$

$$= \frac{4!}{2!\,2!} \times \frac{8!}{3!\,5!}$$

$$= \frac{4\times3\times2!}{2\times1\times2!} \times \frac{8\times7\times6\times5!}{3\times2\times1\times5!}$$

$$= 6 \times 56 = 336 \text{ ways} \quad \text{Ans}$$

2) If one particular clerk should be on the committee this means 2 clerks are to be selected.

No of ways of selection $= {}^7C_2 \times {}^4C_2$

$$= \frac{7!}{2!\,5!} \times \frac{4!}{2!\,2!}$$

$$= \frac{7\times6\times5!}{2\times1\times5!} \times \frac{4\times3\times2!}{2\times1\times2!}$$

$$= 21\times6$$

$$= 126 \text{ ways} \quad \text{Ans}$$

Example - If there are 12 persons in a party and if each two of them shake hands with each other, how many handshakes happen in the party?

Sol - Since handshakes are possible when two persons are there. So, we can say that total number of handshakes is same as the selection of 2 persons.

Therefore, the number of handshakes

$$= {}^{12}C_2$$

$$= \frac{12!}{2!\,10!}$$

$$= \frac{12 \times 11 \times 10!}{2 \times 1 \times 10!} = 66 \text{ ways} \quad \text{Ans}$$

Note - There is only one difference between permutations and combinations which is order.

In permutations the order of arrangement is taken into account but in combinations order doesn't matter.

Ex ab and ba are treated as same in combinations but both are different in permutation.