Replacement Theory

The replacement Theory in operation research is used in the decision making process of replacing a used equipment with a substitute, mostly a new equipment of better usage.

The study of replacement is concerned with situation that arise when some items like machines, men, electric bulbs etc. need replacement due to their inefficiency.

Need of Replacement :- There are following situations, when the replacement of certain items needs to be done -

1) The current equipment has become unusable i.e. it has failed and does not work at all.

2) The current equipment is expected to fail shortly, the probability of which increases over time.

3) The old item has depreciated and requires expensive maintenance.

4) A better design of equipment has been developed due to improvement in technology.

5) when the existing staff gradully disappers due to deaths, retirements etc.

6) In order to increase the competition.

Assumptions of Replacement Theory :-

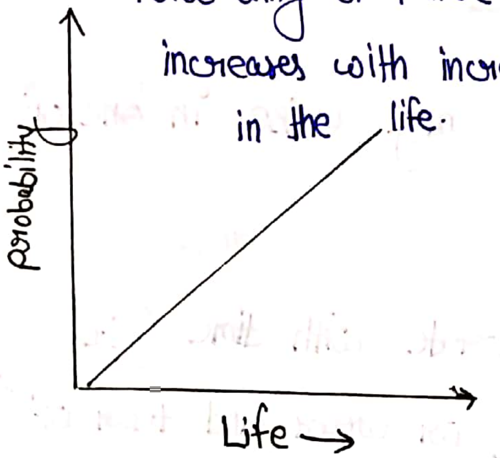
There are following assumptions –

1) The quality of the output remains constant.
2) Replacement and maintance cost remain constant.
3) The operational efficiency of the equipment remains constant.
4) There is no change in technology of the asset under consideration.

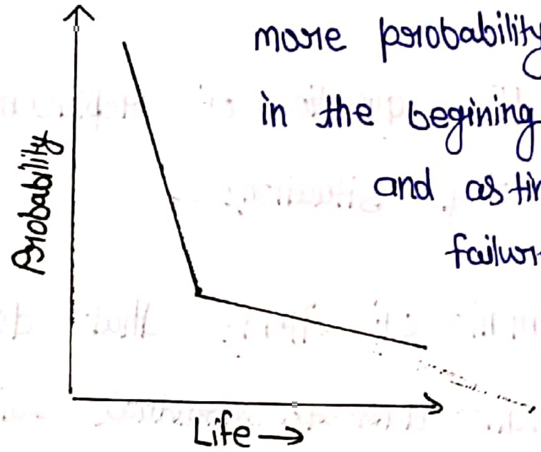
Replacement Decisions :- The replacement decision is taken regarding :

1) Time of replacement
2) How much to replace – what parts to be replaced.
3) How to replace.

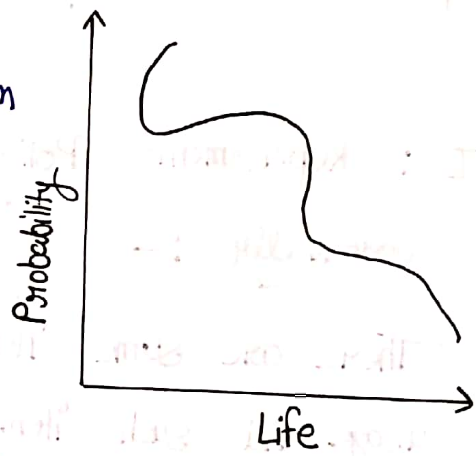
Note –



Probability of failure increases with increase in the life.



more probability of failure in the begining of life and as time passes, failure become less.



equipment fails from random cause like physical shock, irrespective of age.

— There are basically two types of failure.

1) Gradual failure – as life increases that probability of failure also increases.

2) Sudden failure – Those items which fails suddenly.

Types of replacement Problems :- (Replacement Models)

The question of replacement may arise in one of the following situations -

1) Replacement of items that deteriorate with time (i.e. replacement due to gradual failure, or wear and tear of the components of the machines)

2) Replacement of items that fail suddenly.

Type I : Replacement Policy for item that deteriorates Gradually :-

There are some items which deteriorate gradually with usage and such items decline in efficiency over a period of time..

Generally the maintenance cost of certain items always increase gradually with time and a stage comes when the maintenance cost becomes so large that it is better and economical to replace the item with a new one.

This can be further classified into the following types –

(i) Replacement Policy for items whose maintence cost increases with time and value of money remains constant during a Period –

Initially, it is assumed that there is no time value of money.

Let C = The purchase price of the item (Including installation charges)

S = The scrap value of the item at the end of n years (resale value)

and M_t = The maintence cost of the items in year t.

$T(n)$ = Total cost during the period

$A(n)$ = average Total annual cost.

Since we know that –

Total cost = purchase price – Value remaining after n years + Maintenance cost for n years

i.e.

$$T(n) = C - S + \sum_{t=1}^{n} M_t$$

So, average total annual cost = average investment cost + average total running cost during period n.

i.e. $$A(n) = \frac{C-S}{n} + \sum_{t=1}^{n} M_t/n$$

or

$$\boxed{A(n) = \frac{1}{n}\left[C-S+\sum_{t=1}^{n} M_t\right]}$$

where n = number of years the equipment would be in use.

In such case, the optimal replacement period would be the one corresponding to which $A(n)$ would be the minimum.

Example- The cost of an equipment is Rs 62,000 and its scrap value is Rs 2,000. The life of the equipment is 8 years. The maintenance cost for each year are as given -

year	1	2	3	4	5	6	7	8
maintenance cost (Rs)	1000	2000	3500	5000	8000	11000	16000	24000

when the equipment should be replaced?

Sol Given that $C = 62,000$ ₹

$S = 2,000$ ₹

$\therefore$ $C - S = 62,000 - 2,000 = 60,000$ ₹

Now we calculate average ~~total~~ cost per year as follows

Year	Maintenance Cost M_n	Cummlative maintence cost ΣM_n	$C-S$	$C-S+\Sigma M_n = T(n)$	$A_n = \frac{T(n)}{n}$
1	1000	1000	60000	61000	61000
2	2000	3000	60000	63000	31500
3	3500	6500	60000	66500	22167
4	5000	11500	60000	71500	17875
5	8000	19500	60000	79500	15900
6	11000	30500	60000	90500	$\boxed{15083}$ ←
7	16000	46500	60000	106500	15214
8	24000	70500	60000	130500	16312

The average cost is minimum (15083 ₹) at the end of 6th year. Hence, the equipment should be replaced at the end of 6th year.

Example - A manufacture, finds from his past records that the costs per year associated with a machine with a purchase price of 50,000 ₹ are as given below -

year	1	2	3	4	5	6	7	8
Running Cost (in ₹)	15000	16000	18000	21000	25000	29000	34000	40000
Scrap value (₹).	35000	25000	17000	12000	10000	5000	4000	4000

1. Determine the optimum policy.

Sol Here, $C = 50,000$

Since, scrap value is decreasing with time. Therefore, $C-S$ will change with time. Now, average total cost calculated as-

year n	Maintenance Cost M_n(₹)	cummulative maintence cost ΣM_n	S Sr	$C-S$	$T_n = C-S+\Sigma M_n$	$A_n = \frac{T_n}{n}$
1	15000	15000	35000	15000	30,000	30000
2	16000	31000	25000	25000	56000	28000
3	18000	49000	17000	33000	82000	27333.33
4	21000	70000	12000	38000	108000	$\boxed{27000}$
5	25000	95000	10000	40000	135000	$\boxed{27000}$
6	29000	124000	5000	45000	169000	28166.6
7	34000	158000	4000	46000	204000	29142.85
8	40000	198000	4000	46000	244000	30500

The average annual cost is minimum at the end of 4th and 5th year. Hence the machine should be replaced at the end of 5th year.

Ques- The data collected in running a machine, the cost of which is ₹ 30,000 are given below:

Year	1	2	3	4	5
Resale price (₹)	21000	15000	10200	7200	4825
Cost of spares (₹)	2000	2135	2440	2850	3400
Cost of labour (₹)	7000	8000	9000	10500	12500

Determine the optimum period for replacement of the machine.

Sol Here, The cost of spares and labour together will be the maintenance cost.

(ii) Replacement Policy for items whose running cost increases with time but value of money changes with constant rate :-

If the effect of the time value of money is to be considered, then replacement decision analysis must be based upon in equivalent annual cost.

The sum of principle (P) and interest (i) is known as amount A. If the amount (P) is borrowed at a fixed rate of interest i per rupee per year

then one year interest = $i \cdot p$

Every year the interest accured is also added to the principle of perivious year as –

$$P_1 = P + iP = P(1+i)$$

$$P_2 = P_1 + iP_1 = P_1(1+i) = P(1+i)^2$$

similarly, at the end of nth year

$$P(1+i)^n = F \quad \text{(worth of money in future)}$$

So,

$$P = \frac{F}{(1+i)^n}$$

or

$$P = F(1+i)^{-n}$$

Present worth factor –

The above equation can also be written as–

$$P = F \cdot (P.W.F)$$

where $PWF = (1+i)^{-n}$.

The present worth factor is the present value of one rupee spend in n years from time now onwards.

But if $n = 1$ the PWF is given by

$$d = (1+i)^{-1} = \text{discount rate}$$
$$= \text{depreciation value}$$

Let C = Purchase cost of an item

M = Running cost or maintenance cost (annual)

S = Scrap value of item at the end of its life

n = life of items in years.

r = annual interest rate.

Then present worth of the total cost during n years is given by–

$$\boxed{\text{Total Cost} = C + M\,(\text{PWF for } r\% \text{ interest rate for } n \text{ years}) - S\,(\text{PWF for } r\% \text{ interest rate for } n \text{ years})}$$

$$d = \frac{100}{100+r}$$

Ques The yearly cost of 2 machines A and B when money value is neglected is as follows—

Year	1	2	3	4	5
Machine A	1800	1200	1400	1600	1000
Machine B	2800	200	1400	1100	600

Find their cost patterns if money value is 10% per year and hence find which is most economical.

Sol The discounted cost (present worth) at 10% per year for machine A and B is calculated as—

$$d = \frac{1}{1+i} = \frac{1}{1+0.10} = 0.9091$$

Discounted cost of Machine A

Year	Cost	Discount cost at 10% Rate (₹) Present worth
1	1800	$1800 \times 1 = 1800$
2	1200	$1200 \times 0.9091 = 1090.92$
3	1400	$1400 \times (0.9091)^2 = 1157.04$
4	1600	$1600 \times (0.9091)^3 = 1202.14$
5	1000	$1000 \times (0.9091)^4 = 683.04$
		Total = 5933.14

average cost for M/c A $= \frac{5933.14}{5} = 1186.63$ ₹

Discount Cost of Machine B

Year	Discounted cost at 10% Rate (₹) Cost	Present worth
1	2800	$2800 \times 1 = 2800$
2	200	$200 \times 0.9091 = 181.82$
3	1400	$1400 \times (0.9091)^2 = 1157.04$
4	1100	$1100 \times (0.9091)^3 = 826.47$
5	600	$600 \times (0.9091)^4 = 409.82$
		Total = 5375.15

average cost of machine B $= \dfrac{5375.15}{5} = 1075.01$ ₹

Since, the average yearly cost of machine B is less than that for machine A. So, machine B is more economical.

Ques Assume that present value of one rupee to be spent in a years' time is ₹ 0.90 and C = ₹ 6000, capital cost of Equipment. Running costs are given in the table below –

Year	1	2	3	4	5	6	7
Running cost (₹)	1000	1200	1600	2000	2600	3200	4000

When should the machine be replaced?

Ques – A firm pays ₹ 10,000 for its equipments. Their operating and maintenance costs are about ₹ 2500 per year for the first two years and then go up by approximately ₹ 1500 per year. When such equipments be replaced? The discount rate is 10% per year.

Sol

Here $i = 10\% = 0.10$

Therefore, discount rate,

$$d = \frac{1}{1+i} = \frac{1}{1+0.10} = 0.9091$$

Table for weighted average annual cost is as –

$C = 10{,}000$

Year n	Operating & maint cost R_n	Discounted Factor d^{n-1}	Discounted opert & maint. cost $R_n \cdot d^{n-1}$	$\Sigma R_n \cdot d^{n-1}$	Total discounted cost $C + \Sigma R_n d^{n-1} = T_n$	Σd^{n-1}	Weighted avg annual cost $T_n / \Sigma d^{n-1}$
1	2500	1	2500	2500	12500	1	12500
2	2500	0.9091	2272.75	4772.75	14772.75	1.9091	7738.07
3	4000	0.8264	3305.6	8078.35	18078.35	2.7355	6608.79
4	5500	0.7513	4132.15	12210.5	22210.5	3.4868	$\boxed{6369.88}$ minimum
5	7000	0.6830	4781	16991.5	26991.5	4.1698	6473.09
6	8500	0.6209	5277.75	22269.25	32269.25	4.7907	6735.81
7	10,000	0.5645	5645	27914.25	37914.25	5.3552	7079.89
8	11500	0.5132	5901.8	33816.05	43816.05	5.8684	7466.43
9	13,000	0.4665	6149.45	~~39880.45~~ 39965.5	49965.5	6.3349	7887.33

Hence, the optimum length of time to hold the machine before replacing it is 4 years.

Type II :- Replacement of Item That fails Suddenly :-

There are certain items which do not deteriorate with time but fail completely after certain amount of use. These cannot be repaired or maintained to restore their efficiency, hence the only alternatives is to replace them.

Example - Such sudden failure items are electric bulbs, oil seals, resistor in radio etc.

There are situations where the replacement of an individual unit, upon their failure, cost relatively high, whereas if the entire group is replaced then the unit cost works out to be comparatively a much lower. But if the group replacement is done at short intervals, the cost will become high.

Types of Replacement Policies :- In order to minimize the cost associated with sudden failure items, two types of replacement policies are considered —

(a) Individual replacement
(b) Group replacement

Individual Replacement :- Under the individual replacement policy, an item is replaced immediately after its failure. This policy is adopted when the tendency of an item to fail increases with the time it has been in service.

Hence the average replacement cost per year =

Average number of expected replacement per year × Cost of an individual replacement

or

$$\text{Averg. replacement Cost} = \frac{\text{Number of units in operation}}{\text{Expected life an item}} \times \text{Cost of an individual replacement.}$$

Group Replacement :- Group replacement policy is a planned replacement in which all items are replaced at the end of optimal time interval irrespective of whether they have failed or not, plus individually as and when they fail.

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Example - A computer contains 10,000 resistors. when any resistor fails, it is replaced. The cost of replacing a resistor individually is ₹ 1 only. If all the resistors are replaced at the same time, the cost per resistor could be reduced to 35 paisa. The percent surviving at the end of month 't' is given below -

Month	0	1	2	3	4	5	6
% surviving	100	97	90	70	30	15	0

what is the optimum replacement plan ?

Solution- Now, we find out the probability of failure (P_i) during the month t -

month	0	1	2	3	4	5	6
S(t)	100	97	90	70	30	15	0
P_i	—	0·03	0·07	0·20	0·40	0·15	0·15

Let N_i denoted the number of resistors replaced at the end of ith month. Then,

N_0 = number of resistors in the beginning = 10,000

$N_1 = N_0 \cdot P_1 = 10000 \times 0.03 = 300$

$N_2 = N_0P_2 + N_1P_1 = 10000 \times 0.07 + 300 \times 0.03 = 709$

$N_3 = N_0P_3 + N_1P_2 + N_2P_1 = 10000 \times 0.20 + 300 \times 0.07 + 709 \times 0.03$

$= 2042$

$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$

$= 10000 \times 0.40 + 300 \times 0.20 + 709 \times 0.07 + 2042 \times 0.03$

$= 4171$

$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$

$= 10000 \times 0.15 + 300 \times 0.40 + 709 \times 0.20 + 2042 \times 0.07 + 4171 \times 0.03$

$= 2030$

$N_6 = N_0P_6 + N_1P_5 + N_2P_4 + N_3P_3 + N_4P_2 + N_5P_1$

$= 10000 \times 0.15 + 300 \times 0.15 + 709 \times 0.40 + 2042 \times 0.20 + 4171 \times 0.07$

$+ 2030 \times 0.03 = 2590$

Individual Replacement –

— The expected life for each resistor $= \sum x_i P(x_i)$

where x_i = month & $P(x_i)$ = corresponding Probability

So, expected life $= 1\times 0.03 + 2\times 0.07 + 3\times 0.20 + 4\times 0.40 + 5\times 0.15 + 6\times 0.15$

$= 4.02$ month

So, average replacement cost per month =

$$\frac{\text{total Number of resistor}}{\text{expected life}} \times \text{Individual cost}$$

$$= \frac{10000}{4.02} \times 1 = 2487.50 \text{ ₹}$$

Group Replacement -

month	Resistors falling during month	Resistors replaced during month	individual replacement cost	Group replacement cost	Total cost	average cost per month
1	300	300	300	3500	3800	3800
2	709	1009	1009	3500	4509	2254
3	2042	3051	3051	3500	6551	$\boxed{2184}$
4	4171	7222	7222	3500	10722	2680
5	2030	9252	9252	3500	12752	2550
6	2590	11842	11842	3500	15442	2557

From the table, minimum cost per month ₹ 2184 which is obtained by group replacement. So it is optimal to have a group replacement after every 3 month.

Example – A computer has a large number of electronic tubes. They are subjected to given below –

Period	Age of failure (interval of time period)	Probability of failure
1	0–200	0·10
2	201–400	0·26
3	401–600	0·35
4	601–800	0·22
5	801–1000	0·07

If the tubes are group replaced, the cost of replacement is ₹ 15 per tube. Group replacement can be done at a fixed interval in the night shift when the computer is not normally used. Replacement of individual tubes which fail in service cost ₹ 60 per tube. How frequently should the tubes be replaced?

Sol – Let N_i represent no of tubes replaced at the end of period i.

Let N_0 = Number of tubes in the system = 1000

$N_1 = N_0P_1 = 1000 \times 0{\cdot}10 = 100$

$N_2 = N_0P_2 + N_1P_1 = 1000 \times 0{\cdot}26 + 100 \times 0{\cdot}10 = 270$

$N_3 = N_0P_3 + N_1P_2 + N_2P_1 = 1000 \times 0{\cdot}35 + 100 \times 0{\cdot}26 + 270 \times 0{\cdot}10$

$= 403$

$$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$$

$$= 1000 \times 0.22 + 100 \times 0.35 + 270 + 0.26 + 403 \times 0.10$$

$$= 365$$

$$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$$

$$= 1000 \times 0.07 + 100 \times 0.22 + 270 \times 0.35 + 403 \times 0.26 + 365 \times 0.10 = 328$$

Individual replacement -

$$\text{Expected life} = \sum x_i P(x_i)$$

$$= 1 \times 0.10 + 2 \times 0.26 + 3 \times 0.35 + 4 \times 0.22 + 5 \times 0.07$$

$$= 2.90 \text{ periods}$$

∴ cost of individual replacement per period

$$= \frac{\text{Total Number of tubes}}{\text{Expected life}} \times \text{Individual cost}$$

$$= \frac{1000}{2.90} \times 60 = 20,700 ₹$$

Group Replacement -

Period	Tubes falling during Period i	tubes replaced upto period i	Individual Repl. cost	Group Replac. Cost	Total Cost	average Cost
1	100	100	6000	15000	21000	21000
2	270	370	22,200	15000	37200	$\boxed{18600}$
3	403	773	46380	15000	61380	20460
4	365	1138	68280	15000	83280	20820
5	328	1466	87960	15000	102960	20592

Here, minimum average cost is ₹ 18600 which is obtained by group replacement.

Hence, the optimal group replacement policy is used to replace all tubes at the end of 2nd intervals.