Numerical Integration

Consider a definite integral as –

$$I = \int_a^b f(x)\,dx$$

The computation of a definite integral I from a set of numerical values of integral is called numerical integration. If the integrand is a function of single variable, the process is called numerical quadrature.

Methods for Numerical Integration :– There are various methods for the numerical integration –

1) Trapezoidal Rule
2) Simpson's 1/3rd rule
3) Simpson's $\frac{3}{8}$th rule

Trapezoidal Rule

Let $I = \int_a^b f(x)\,dx$ be the integral

where $y = f(x)$.

Let $f(x)$ be a given for certain equally distant values of arguments say $x_0, x_0+h, x_0+2h, \ldots$ so that $y_0, y_1, y_2, \ldots$ value found. i.e.

$$a = x_0 \xrightarrow{h} x_1 \xrightarrow{h} x_2 \xrightarrow{h} x_3 \cdots\cdots x_n = b$$

$$y_0 \qquad y_1 \qquad y_2 \qquad y_3 \cdots\cdots y_n$$

Let the range $(b-a)$ be divided into n equal parts, each of which is of width h i.e. $b-a = n \cdot h$

According to Trapezoidal Rule –

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots\cdots y_{n-1})\right]$$

or $I = \frac{h}{2}$ [(sum of first & last values) + 2 × (sum of remaining values)]

Example - Evaluate $\int_{-3}^{3} x^4\, dx$ by using Trapezoidal rule.

Sol

Let $y(x) = x^4$ and interval length $= b-a = 6$

So, we divide 6 equal interval with $h = \frac{6}{6} = 1$

we create the table such that

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal Rule,

$$I = \frac{h}{2}\left[(y_0+y_6) + 2\left(y_1+y_2+y_3+y_4+y_5\right)\right]$$

$$= \frac{1}{2}\left[(81+81) + 2(16+1+0+1+16)\right]$$

$$= \frac{1}{2}\left[162+68\right]$$

$$= \frac{1}{2}(230)$$

$$= 115$$ **Ans**

Simpson's 1/3 rd Rule

It is also called simpson's rule. Consider the definite integral as –

$$I = \int_a^b f(x)\,dx \quad \text{where } y = f(x)$$

again,

$$a = x_0 \xrightarrow{h} x_1 \xrightarrow{h} x_2 \xrightarrow{h} x_3 \; \cdots \cdots \; x_n = b$$

$$y_0 \quad y_1 \quad y_2 \quad y_3 \; \cdots \cdots \; y_n$$

argument x is equally distance with h.

By Simpson's or simpson's 1/3 rd rule –

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + \cdots)\right]$$

or

$$I = \frac{h}{3}\left[(\text{sum of 1st and last value}) + 4(\text{sum of odd terms}) + 2(\text{sum of even terms})\right]$$

Example – By using simpson's rule solve $\int_0^1 \frac{1}{1+x} dx$,

taking $h = 0.2$.

Sol

Here, $y = \frac{1}{1+x}$ and $h = 0.2$

tables for values are as –

x	0	0.2	0.4	0.6	0.8	1
y	1	0.8333	0.7142	0.625	0.555	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

using simpson's rule is –

$$I = \frac{h}{3}\left[(y_0+y_5) + 4(y_1+y_3) + 2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1+0.5)+4(0.8333+0.625)+ 2(0.7142+0.555)\right]$$

$$= \frac{0.2}{3}\left[1.5+5.8332+2.5384\right]$$

$$= 0.6581 \quad \underline{\text{Ans}}$$

Simpson's 3/8 th Rule

Consider the definite integral as –

$$I = \int_a^b f(x)\,dx \quad , \quad \text{where } y = f(x)$$

Let values are –

$$a = x_0 \xrightarrow{h} x_1 \xrightarrow{h} x_2 \xrightarrow{h} x_3 \cdots\cdots x_n = b$$

$$y_0 \quad y_1 \quad y_2 \quad y_3 \cdots\cdots y_n$$

Values of x are equally distant

According to simpson's 3/8 th Rule –

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + \cdots) + 3(y_1 + y_2 + y_4 + \cdots)\right]$$

OR

$$I = \frac{3h}{8}\Big[(\text{sum of first and last terms}) + 2\times(\text{sum of terms which is multiple of } 3) + 3(\text{sum of remaining terms})\Big]$$

Example – Evaluate $\int_0^3 \frac{1}{1+x}\,dx$ with $n=6$ by using simpson's 3/8 rule.

Sol –

Here, $y = \frac{1}{1+x}$ and $h = \frac{b-a}{h} = \frac{3-0}{6} = 0.5$

Now, values are calculated as –

x	0	0.5	1	1.5	2	2.5	3
y	1	0.6667	0.5	0.4	0.333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By using simpson's $\frac{3}{8}$ Rule

$$I = \frac{3h}{8}\left[(y_0+y_6) + 2y_3 + 3\,(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3(0.5)}{8}\left[(1+0.25) + 2(0.4) + 3\,(0.6667+0.5+0.333+0.2857)\right]$$

$$= \frac{1.5}{8}\left[1.25 + 0.8 + 5.3562\right]$$

$$= 1.3886 \quad \text{Ans}$$

Note -

1) simpson's 1/3 rd rule is generally used when number of total x terms is even.

2) simpson's 3/8 th rule is preferred if number of terms are multiple of three.

Example - Evaluate $\int_0^6 x^2 dx$ by using all method. Taking $h=1$.

Sol - Let $I = \int_0^6 x^2 dx$

Here, $y = x^2$, $h = 1$

Now, we create table -

x	0	1	2	3	4	5	6
y	0	1	4	9	16	25	36
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

using Trapezoidal rule -

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(0+36)+2(1+4+9+16+25)\right]$$

$$I = \frac{1}{2}\left[36 + 110\right] = 73 \quad \text{Ans}$$

By using simpson's 1/3 rule

$$I = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(0+36) + 4(1+9+25) + 2(4+16)\right]$$

$$= \frac{1}{3}\left[36 + 140 + 40\right]$$

$$= 72 \quad \text{Ans}$$

By using simpson's 3/8 rule-

$$I = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left[(0+36) + 2(9) + 3(1+4+16+25)\right]$$

$$= \frac{3}{8}\left[36 + 18 + 138\right]$$

$$= 72 \quad \text{Ans}$$