Numerical Methods

BCA 4sem

Solution of Linear Equation

Let us consider equation of linear equation having n unknowns as –

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$$

Generally, these equations is solved by matrixces –

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{B}$$

then, the equations can be written as –

$$AX = B$$

and if $|A| \neq 0$ (i.e. A is non singular)

then $\boxed{X = A^{-1}.B}$

In numerical methods, for solution of linear equation most important methods are included which is as follows–

1) Gauss's Elimination Method

2) Gauss's Seidel Iterative Method

Gauss Elimination Method

Gauss elimination method is named after German mathematician and scientist Carl Friedrich Gauss.

Consider the following system of linear equation–

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{13}x_3 = b_1 \quad \text{--- (I)}$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{23}x_3 = b_2 \quad \text{--- (II)}$$

$$a_{31}x_1 + a_{32}x_2 + \dots + a_{33}x_3 = b_3 \quad \text{--- (III)}$$

There are following steps are used to solve the system of linear equation–

Step-1 suppose that coefficient of x_1 in the first equation is not equal to zero i.e. $a_{11} \neq 0$.

Step-2 Elimination of x_1– Now, with the help of equation first we can eliminate x_1 from 2nd & 3rd equation in the following way.

(a) Divide 1st equation by a_{11}, so that coefficient of x_1 becomes 1 then subtract a_{21} times of first equation from second equation.

(b) Similarly after dividing 1st equation by a_{11}, subtract a_{31} times of equation 1 from second equation.

The new system of linear equations becomes

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + \cdots + a'_{23}x_3 = b'_2$$
$$a'_{32}x_2 + \cdots + a'_{33}x_3 = b'_3$$

Step-3 Elimination of x_2 – Now suppose a'_{22} is not equal to zero then eliminate x_2 from 3rd equation by using 2nd equation.

Divide second equation by a'_{22} and subtract a'_{32} times of second equation to third equation. Then the new equations are in the form of –

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a''_{33}x_3 = b''_3$$

Step-4 Back Substitution – Now we can find out x_3 from third equation and after that by back substitution x_1 and x_2.

Note – There are two cases in solving linear equation by using Gauss elimination method –

with pivot = No change

without pivot = interchange the equation if necessary (perform row interchange with the coefficient of largest absolute value variable first)

If question is not described any condition then we generally used without pivot case.

Example - Solve the following system of equations using the Gauss elimination method.

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

Sol since the coefficient of x in eq (ii) has the largest value. So, we interchange the equations and get

pivot eq^n → $8x - 3y + 2z = 20$

$$4x + 11y - z = 33$$
$$2x + y + 4z = 12$$

or

$$\left[\begin{array}{ccc:c} 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \\ 2 & 1 & 4 & 12 \end{array}\right]$$

Now, we solve the equations. The steps are as –

1) **Elimination of x** – Now, we eliminate x from eqⁿ (II) & (III) with the help of eq (I). So,

subtract $\frac{4}{8} = \frac{1}{2}$ times of eq (i) from the eq (ii) and also subtract $\frac{2}{8} = \frac{1}{4}$ times of eq (i) from the eq (iii), we get

$$8x - 3y + 2z = 20$$

pivot $\rightarrow$ $$\frac{25}{2}y - 2z = 23$$

$$\frac{7}{4}y + \frac{7}{2}z = 7$$

$$\left[\begin{array}{ccc|c} 8 & -3 & 2 & 20 \\ 0 & \frac{25}{2} & -2 & 23 \\ 0 & \frac{7}{4} & \frac{7}{2} & 7 \end{array}\right]$$

2) **Elimination of y** – since, eqⁿ (II) has the largest coefficient of y. Hence we take 2nd equation as the pivot equation.

subtract $\frac{7}{4} \Big/ \frac{25}{2} = \frac{7}{50}$ times pivot equations from the eqⁿ (III), we get

$$8x - 3y + 2z = 20$$

$$\frac{25}{2}y - 2z = 23$$

$$\frac{189}{50}z = \frac{189}{50}$$

$$\left[\begin{array}{ccc|c} 8 & -3 & 2 & 20 \\ 0 & \frac{25}{2} & -2 & 23 \\ 0 & 0 & \frac{189}{50} & \frac{189}{50} \end{array}\right]$$

3) **Back Substitution** – From taking end to first equation

$$z = 1$$

$$y = \frac{2}{25}(23 + 2z) = \frac{2}{25} \times 25 = 2$$

$$x = \frac{1}{8}(20 + 3y - 2z) = \frac{1}{8}(20 + 6 - 2) = \frac{24}{8} = 3$$

So, $x = 3$, $y = 2$ and $z = 1$ Ans

Example - Solve the following system of equations using Gauss elimination method with pivoting -

$$2x_1 + 3x_2 + 4x_3 = 20$$
$$4x_1 + 2x_2 + 3x_3 = 17$$
$$x_1 + 4x_2 + 2x_3 = 17$$

Sol - Since, we solve given equation, with pivoting then we don't interchange any equation. The steps to solve the equation are as follows -

1) **Elimination of x** - In this step, we eliminate variable x_1 from eq (ii) & (iii) with the help of eq (i).

Now, subtract $\frac{4}{2} = 2$ times of eq (i) from eq (ii) and also subtract $\frac{1}{2}$ times of eq (i) from eq (iii), and we get

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 20 \\ -4x_2 - 5x_3 &= -23 \\ \frac{5}{2}x_2 + 0 \cdot x_3 &= 7 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc:c} 2 & 3 & 4 & 20 \\ 0 & -4 & -5 & -23 \\ 0 & \frac{5}{2} & 0 & 7 \end{array}\right]$$

There is no need of elimination of x_2 from eq (iii). So, we direct perform back substitution.

Back Substitution - In this step, first we consider last equation, then second equation and after that first equation. So, we have -

$$x_2 = 7 \times \frac{2}{5} = \frac{14}{5}$$

and $$x_3 = \frac{1(-23+4x_2)}{-5}$$

or $$x_3 = -\frac{1}{5}\left(-23+4\times\frac{14}{5}\right) = \frac{59}{25}$$

similarly, $$x_1 = \frac{1}{2}\left(20-3x_2-4x_3\right)$$

$$= \frac{1}{2}\left[20-3\times\frac{14}{5}-4\times\frac{59}{25}\right] = \frac{27}{25}$$

So, the required solution is -

$$x_1 = \frac{27}{25}, \quad x_2 = \frac{14}{5}, \quad x_3 = \frac{59}{25}$$

Ques - Solve the following system of equations, using Gauss elimination method -

$$x+y+z = 6.6$$
$$x-y+z = 2.2$$
$$x+2y+3z = 15.2$$

Gauss Seidel Method

This method is also used to solve the system of linear equations. Consider the following system of linear equations –

$$a_1x + b_1y + c_1z = d_1 \quad \text{—— (I)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{—— (II)}$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{—— (III)}$$

Before apply Gauss Seidel iterative method, we check some particular conditions. These conditions are –

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

These conditions are called Diagonally dominant Condition.

If given equations not satisfied diagonally dominant Condition then we interchange the given equations to each other until it satisfy the diagonally dominant condition.

Now, we solve for variables x, y, z in terms of the other variables. i.e.

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \quad \text{———— (A)}$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \text{———— (B)}$$

$$\& \quad z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \quad \text{———— (C)}$$

we start with the initial values of y and z i.e. $y^{(0)}$ & $z^{(0)}$ and get $x^{(1)}$ from the eqn (A).

while using the eqn (B), we calculate $y^{(1)}$ by putting $x^{(1)}$ and $z^{(0)}$.

and then using $x^{(1)}$ and $y^{(1)}$ values, we calculate $z^{(1)}$ from the equation (C).

— This process of iteration is continued until the convergence is assured.

Example – Solve by Gauss – Seidel method, the following system of equations —

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

Sol – First we check diagonally dominant conditions as –

$$|28| > |4| + |-1| \quad (\text{true})$$

$$|3| \ngtr |1| + |10| \quad (\text{false})$$

$$|4| \ngtr |2| + |17| \quad (\text{false})$$

Since, the given system is not diagonally dominant. Hence, we interchange the equations such that they satisfy the diagonally dominant condition.

So,

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

Now, we apply Gauss Seidel iterative method as –

From the above equations –

$$x = \frac{1}{28}[32 - 4y + z] \quad \text{——— (I)}$$

$$y = \frac{1}{17}[35 - 2x - 4z] \quad \text{——— (II)}$$

$$z = \frac{1}{10}[24 - x - 3y] \quad \text{——— (III)}$$

First Approximation – putting $y = z = 0$ in eq (i), we get

$$x^{(1)} = \frac{1}{28}(32) = 1.1429$$

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putting $x = 1.1429$, $z = 0$ in eq (ii), we have

$$y^{(1)} = \frac{1}{17}\left[35 - 2(1.1429) + 0\right] = 1.9244$$

putting $x = 1.1429$, $y = 1.9244$ in eq (iii), we get

$$z^{(1)} = \frac{1}{10}\left[24 - 1.1429 - 3(1.9244)\right] = 1.7084$$

Second Approximation -

putting $y = 1.9244$, $z = 1.7084$ in eq (i), we get

$$x^{(2)} = \frac{1}{28}\left[32 - 4(1.9244) + 1.7084\right] = 0.9289$$

putting $x = 0.9289$, $z = 1.7084$ in eq (ii), we get

$$y^{(2)} = \frac{1}{17}\left[35 - 2(0.9289) - 4(1.7084)\right] = 1.5476$$

putting $x = 0.9289$, $y = 1.5476$ in eq (iii), we have

$$z^{(2)} = \frac{1}{10}\left[24 - 0.9289 - 3(1.5476)\right] = 1.8428$$

Third Approximation -

$$x^{(3)} = \frac{1}{28}\left[32 - 4(1.5476) + 1.8428\right] = 0.9876$$

$$y^{(3)} = \frac{1}{17}\left[35 - 2(0.9876) - 4(1.8428)\right] = 1.5090$$

$$z^{(3)} = \frac{1}{10}\left[24 - 0.9876 - 3(1.5090)\right] = 1.8485$$

Fourth Approximation –

$$x^{(4)} = \frac{1}{28}\left[32-4(1.5090)+1.8485\right] = 0.9933$$

$$y^{(4)} = \frac{1}{17}\left[35-2(0.9933)-4(1.8485)\right] = 1.5070$$

$$z^{(4)} = \frac{1}{10}\left[24-0.9933-3(1.5070)\right] = 1.8485$$

Fifth Approximation –

$$x^{(5)} = \frac{1}{28}\left[32-4(1.5070)+1.8485\right] = 0.9935$$

$$y^{(6)} = \frac{1}{17}\left[35-2(0.9935)-4(1.8485)\right] = 1.5069$$

$$z^{(6)} = \frac{1}{10}\left[24-0.9935-3(1.5069)\right] = 1.8485$$

Hence, solution is –

$$\left.\begin{aligned} x &= 0.9935 \\ y &= 1.5069 \\ z &= 1.8485 \end{aligned}\right\} \quad \underline{\underline{\text{Ans}}}$$

Example - Solve the following system of equation using Gauss Seidel Iterative method (four iterations)

$$20x_1 + 5x_2 - 2x_3 = 14$$
$$3x_1 + 10x_2 + x_3 = 17$$
$$x_1 - 4x_2 + 10x_3 = 23$$

Sol Since,

$$|20| > |5| + |-2| \quad \text{(true)}$$
$$|10| > |3| + |1| \quad \text{(true)}$$
$$|10| > |1| + |-4| \quad \text{(true)}$$

the given system of equations are diagonally dominant.

So, we apply the Gauss Seidel iterative method as—

From the above equations—

$$x_1 = \frac{1}{20}\left[14 - 5x_2 + 2x_3\right] \quad \text{—— (i)}$$

$$x_2 = \frac{1}{10}\left[17 - 3x_1 - x_3\right] \quad \text{—— (ii)}$$

$$x_3 = \frac{1}{10}\left[23 - x_1 + 4x_2\right] \quad \text{—— (iii)}$$

First Approximation— putting $x_2 = x_3 = 0$ in eq (i), we have

$$x_1^{(1)} = \frac{1}{20}\left[14\right] = 0.7$$

putting $x_1 = 0.7$, $x_3 = 0$ in eq^n (ii), we have

$$x_2^{(1)} = \frac{1}{10}[17 - 3(0.7) - 0] = 1.49$$

putting $x_1 = 0.7$, $x_2 = 1.49$ in eq^n (iii), we have

$$x_3^{(1)} = \frac{1}{10}[23 - 0.7 + 4(1.49)] = 2.826$$

2nd Approximation - putting $x_2 = 1.49$, $x_3 = 2.826$ in eq (i),

$$x_1^{(2)} = \frac{1}{20}[14 - 5(1.49) + 2(2.826)] = 0.6101$$

putting $x_1 = 0.6101$, $x_3 = 2.826$ in eq (ii), we have

$$x_2^{(2)} = \frac{1}{10}[17 - 3(0.6101) - 2.826] = 1.2344$$

putting $x_1 = 0.6101$, $x_2 = 1.2344$ in eq (iii), we have

$$x_3^{(2)} = \frac{1}{10}[23 - 0.6101 + 4(1.2344)] = 2.7328$$

3rd Approximation - Similarly using the calculated values, we find the next approximation -

$$x_1^{(3)} = \frac{1}{20}[14 - 5(1.2344) + 2(2.7328)] = 0.6647$$

$$x_2^{(3)} = \frac{1}{10}\left[17 - 3(0.6647) - 2.7327\right] = 1.2273$$

$$x_3^{(3)} = \frac{1}{10}\left[23 - 0.6647 + 4(1.2273)\right] = 2.7245$$

4th Approximation —

$$x_1^{(4)} = \frac{1}{20}\left[14 - 5(1.2273) + 2(2.7245)\right] = 0.6656$$

$$x_2^{(4)} = \frac{1}{10}\left[17 - 3(0.6656) - 2.7245\right] = 1.2279$$

$$x_3^{(4)} = \frac{1}{10}\left[23 - 0.6656 + 4(1.2279)\right] = 2.7246$$

Hence,

$$x_1 = 0.6656, \quad x_2 = 1.2279 \quad \& \quad x_3 = 2.7246$$

Gauss Jacobi Method

This method is same as Gauss Seidel method only difference is that Gauss Jacobi method used all the values of previous approximation rather than the current calculated value.

Example - Solve the following system of linear equation by using Gauss Jacobi method -

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Sol Since,

$$|27| > |6| + |-1|$$

$$|15| > |6| + |2|$$

$$|54| > |1| + |1|$$

So, the given equations are diagonally dominant.

Now, we apply Gauss Jacobi method.

The given equations are arranged as -

$$x = \frac{1}{27}[85 - 6y + z] \quad \text{------ (I)}$$

$$y = \frac{1}{15}[72 - 6x - 2z] \quad \text{------ (II)}$$

$$z = \frac{1}{54}[110 - x - y] \quad \text{------ (III)}$$

1st approximation - putting $x = y = z = 0$ in eqn. (i), (ii) & (iii), we get

$$x^{(1)} = \frac{85}{27} = 3.1481$$

$$y^{(1)} = \frac{1}{15}[72] = 4.8$$

$$z^{(1)} = \frac{1}{54}[110] = 2.037$$

2nd approximation -

Now put $x = 3.1481$, $y = 4.8$ & $z = 2.037$ in eqn (i), (ii) & (iii) & we have

$$x^{(2)} = \frac{1}{27}\left[85 - 6(4.8) + 2.037\right] = 2.1569$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6(3.1481) - 2(2.037)\right] = 3.2691$$

$$z^{(2)} = \frac{1}{54}\left[110 - 3.1481 - 4.8\right] = 1.8899$$

3rd Approximation -

$$x^{(3)} = \frac{1}{27}\left[85 - 6(3.2691) + 1.8899\right] = 2.4917$$

$$y^{(3)} = \frac{1}{15}\left[72-6(2.1569)-2(1.8899)\right] = 3.6852$$

$$z^{(3)} = \frac{1}{54}\left[110-2.1569-3.2691\right] = 1.9366$$

4th approximation –

$$x^{(4)} = \frac{1}{27}\left[85-6(3.6852)+1.9366\right] = 2.4009$$

$$y^{(4)} = \frac{1}{15}\left[72-6(2.4917)-2(1.9366)\right] = 3.5451$$

$$z^{(4)} = \frac{1}{54}\left[110-2.4917-3.6852\right] = 1.9227$$

Hence, $x = 2.4009$, $y = 3.5451$ & $z = 1.9227$

ill – Conditioned Systems — A system of linear equation is called ill conditioned system, if very small change in the coefficients of the system may produce large changes in the solution.

A system of equations is considered to be well conditioned if a small change in the coefficient of variable or a small change in the right hand side of equation produce a small change in the solution.

Example - Show that the following system of equations is ill conditioned.

$$x + 5y = 17$$

$$1.5x + 7.501y = 25.503$$

Sol These equations can be easily calculated and we get $x = 2$ and $y = 3$

Now, let there be a very small change (0.003) in the second equation such that

$$x + 5y = 17$$

$$1.5x + 7.501y = 25.500$$

The solution of these equation is -

$$x = 17, \quad y = 0$$

So, solution is changed from $(2, 3)$ to $(17, 0)$.

Hence, these equations are called ill conditioned.