Numerical Methods BCA 4th Sem

Solution of Differential Equation

Solution of differential Equation is one of the major problem of numerical analysis. The classical initial value problem is to find a function $y = f(x)$ which satisfy the first order differential equation –

$$\frac{dy}{dx} = f(x,y)$$

with initial condition $f(x_0) = y_0$

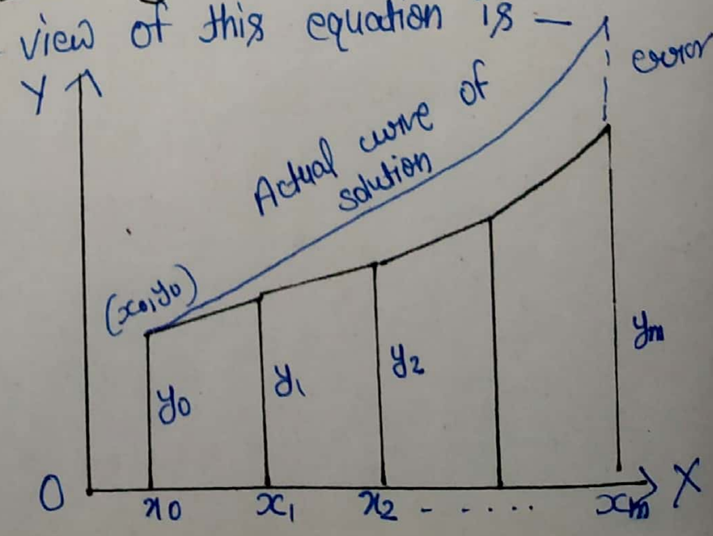
Methods to solve Differential Equation :-

There are various methods to solve the differential equations. Some of the methods are defined as –

1) **Euler's Method** :- Let us consider the general first order differential equation as –

$$\frac{dy}{dx} = f(x,y) \quad \text{where } y(x_0) = y_0 \qquad (1)$$

Now, the graphical point of view of this equation is –



suppose, we wish to find $y_1, y_2, \ldots y_m$ for equally distance value $x_1, x_2, \ldots x_m$

where, $x_m = x_0 + mh$

$(m = 1, 2, \ldots h)$

Now, we uses the property that in a small interval, a curve is nearly a straight line. Thus, in the interval x_0 to x_1, one approximate the curve by the tanget at the point (x_0, y_0)

Therefore, the equation of the tanget at the point (x_0, y_0) is –

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} \cdot (x - x_0)$$

or $$y - y_0 = f(x_0, y_0)(x - x_0) \quad \text{(from eq (i))}$$

or $$y = y_0 + f(x_0, y_0)(x - x_0)$$

Hence, the value of y corresponding to $x = x_1$ is

$$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$$

or $$y_1 = y_0 + h f(x_0, y_0) \quad \left(\text{since } x_m = x_0 + mh,\ m = 1, 2, \ldots\right)$$

Similarly, approximating the curve in the next interval $[x_1, x_2]$ by a line through (x_1, y_1) is

$$y_2 = y_1 + h f(x_1, y_1)$$

Proceding on, in general it can be shown that

$$\boxed{y_{m+1} = y_m + h f(x_m, y_m)}.$$

This is Euler method or Euler Cauchy method.

Example - Use Euler's method to solve the following differential equation $\frac{dy}{dx} = 1-y$ with $y(0) = 0$ in the range $0 \leq x \leq 0.3$

Sol Given that $\frac{dy}{dx} = 1-y, \quad y(0) = 0$

Here, $f(x,y) = 1-y, \quad x_0 = 0, \quad y_0 = 0$

& let $= h = 0.1$

Now, we have to find out the solution at

$x = 0.1, \; 0.2 \; \& \; 0.3$

According to Euler's method -

$$y_{m+1} = y_m + hf(x_m, y_m)$$

Now,

$$y(0.1) = y_1 = y_0 + hf(x_0, y_0)$$
$$= 0 + 0.1(1-0) = 0.1$$

$$y(0.2) = y_2 = y_1 + hf(x_1, y_1)$$
$$= 0.1 + 0.1 f(0.1, 0.1)$$
$$= 0.1 + 0.1(1-0.1) = 0.19$$

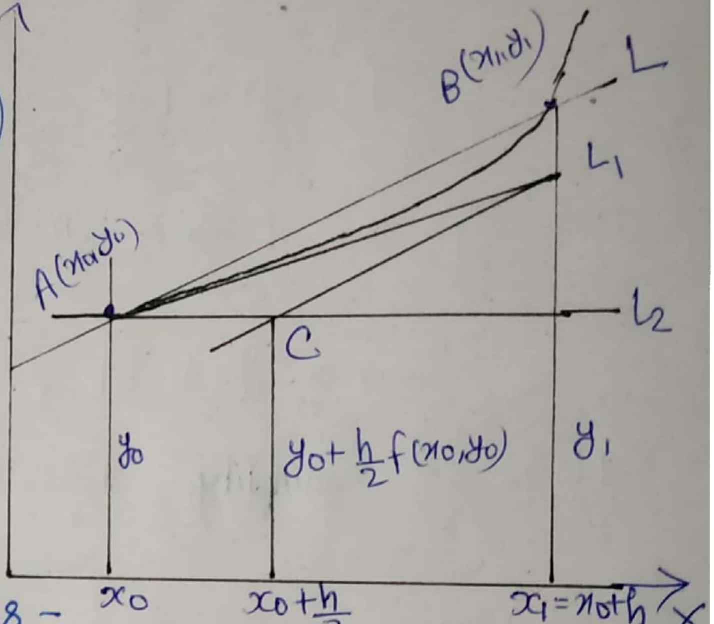
& $$y(0.3) = y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.19 + 0.1 f(0.2, 0.19)$$
$$= 0.19 + 0.1(1-0.19) = 0.271$$

Modified Euler's Method :-

In this method, the curve in the interval (x_0, x_1) where $x_1 = x_0 + h$ is approximated by the line through (x_0, y_0) with slope

$$f\left(x_0+\frac{h}{2},\ y_0+\frac{h}{2}f(x_0,y_0)\right)$$

i.e. the slope at the middle point whose distance of a point from the y-axis is the average of x_0 and x_1 i.e. $x_0+\frac{h}{2}$



So, The equation for line L is -

$$y-y_0 = (x-x_0)\left\{f\left(x_0+\frac{h}{2},\ y_0+\frac{h}{2}f(x_0,y_0)\right)\right\}$$

putting $x = x_1$, we get

$$y_1-y_0 = (x_1-x_0)\left\{f\left(x_0+\frac{h}{2},\ y_0+\frac{h}{2}f(x_0,y_0)\right)\right\}$$

$$y_1 = y_0 + hf\left(x_0+\frac{h}{2},\ y_0+\frac{h}{2}f(x_0,y_0)\right)$$

proceeding in the same way, it can be shown that

$$\boxed{y_{m+1} = y_m + hf\left(x_m+\frac{h}{2},\ y_m+\frac{h}{2}f(x_m,y_m)\right)}$$

Example - Solve $\frac{dy}{dx} = 1-y$, $y(0) = 0$ in the range $0 \leq x \leq 0.3$ using modified Euler's method by choosing $h = 0.1$

Sol According to the question,

$$f(x,y) = 1-y, \quad x_0 = 0, \quad y_0 = 0 \ \& \ h = 0.1 \quad \text{—— (i)}$$

According to Euler's modified method -

$$y_{m+1} = y_m + hf\left\{x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\right\}$$

Now, we simplify these equation by putting function -

So,

$$y_{m+1} = y_m + h\left\{1 - \left[y_m + \frac{h}{2} f(x_m, y_m)\right]\right\} \quad (\text{from eq (i)})$$

$$= y_m + h\left\{1 - y_m - \frac{h}{2}(1-y_m)\right\}$$

$$= y_m + h\left\{\left(1-\frac{h}{2}\right)(1-y_m)\right\}$$

$$= y_m + 0.1\left(1 - \frac{0.1}{2}\right)(1-y_m) \quad (\because h = 0.1)$$

$$= y_m + 0.095(1-y_m)$$

$$= 0.095 - 0.095y_m + y_m$$

$$y_{m+1} = 0.095 + 0.905\, y_m \quad \text{—— (ii)}$$

putting the value of m successively in eq (ii) & get required solution.

Hence,

$$y(0.1) = y_1 = 0.095 + 0.905 y_0$$
$$= 0.095 + 0.905 \times 0 = 0.095$$

$$y(0.2) = y_2 = 0.095 + 0.905 y_1$$
$$= 0.095 + 0.905 \times 0.095 = 0.1809$$

$$y(0.3) = y_3 = 0.095 + 0.905 y_2$$
$$= 0.095 + 0.905 \times 0.1809 = 0.2587$$

Example - Solve $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ and find $y(0.1)$, $y(0.2)$ by using (i) Euler's method and
(ii) Modified Euler's method

Solve - Given that $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$

Here, $f(x,y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 1$ & $h = 0.1$

1) By Euler's method -

$$y_{m+1} = y_m + hf(x_m, y_m)$$

$$\therefore \quad y(0.1) = y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.1\, f(0, 1)$$
$$= 1 + 0.1\,(0^2 + 1^2) = 1.1$$

$$\& \quad y(0.2) = y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.1 + 0.1\,((0.1)^2 + (1.1)^2) = 1.222$$

2) using modified Euler's method -

$$y_{m+1} = y_m + h\left[f\left(x_m+\frac{h}{2},\ y_m+\frac{1}{2}hf(x_m, y_m)\right)\right]$$

$$\therefore\ y(0.1) = y_1 = y_0 + hf\left[x_0+\frac{h}{2},\ y_0+\frac{1}{2}hf(x_0, y_0)\right] \quad \text{—— (i)}$$

$$\because\ y_0+\frac{1}{2}hf(x_0,y_0) = y_0+\frac{h}{2}(x_0^2+y_0^2)$$

$$= 1+\frac{0.1}{2}(0+1^2) = 1.05$$

from eq (i),

$$y_1 = 1+0.1\,f\left(0+\frac{0.1}{2},\ 1.05\right)$$

$$= 1+0.1\left[(0.05)^2+(1.05)^2\right] = 1.1105$$

& $x_1 = x_0+h = 0+0.1 = 0.1$

again

$$y(0.2) = y_2 = y_1+hf\left[x_1+\frac{h}{2},\ y_1+\frac{1}{2}h\,f(x_1,y_1)\right] \quad \text{—— (ii)}$$

$$\therefore\ y_1+\frac{h}{2}f(x_1,y_1) = 1.1105+\frac{0.1}{2}f(0.1,\ 1.1105)$$

$$= 1.1105+0.05\left[(0.1)^2+(1.1105)^2\right]$$

$$= 1.1726$$

from eq (ii),

$$y_2 = 1.1105+0.1\,f\left(0.1+\frac{0.1}{2},\ 1.1726\right)$$

$$= 1.1105+0.1\left[(0.15)^2+(1.1726)^2\right]$$

$$= 1.25026$$

Picard's Method :- Consider an differential Equation as -

$$\frac{dy}{dx} = f(x,y) \quad \text{with} \quad y(x_0) = y_0$$

it can be written as -

$$dy = f(x,y)\,dx$$

Integrating this equation, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow \quad y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow \quad y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \quad \text{——— (1)}$$

This is called an integral equation. It is solved by the method of successive approximation in which 1st approximation is obtained by substituting y_0 for y on the right hand side of eq (i). So,

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

Now, put $y = y_2$ then from eq (i)

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$$

proceeding on the similar way, we can obtain $(m+1)^{th}$ approximation from the mth approximation as-

$$\boxed{y_{m+1} = y_0 + \int_{x_0}^{x} f(x, y_m)\, dx}$$

Example - Find the solution of $\frac{dy}{dx} = x+y$ with $y(0)=1$ by using Picard's method. (upto 5th approximation)

Sol - Since, $f(x,y) = x+y$, $x_0=0$ & $y_0=1$

According to Picard's method, the integral Equation is —

$$y_{m+1} = y_0 + \int_{x_0}^{x} f(x, y_m)\, dx$$

or $$y_{m+1} = 1 + \int_{0}^{x} f(x, y_m)\, dx \quad (\because x_0=0,\ y_0=1)$$

1st approximation -

$$y_1 = 1 + \int_0^x f(x, y_0)\, dx$$

or $$y_1 = 1 + \int_0^x (x+1)\, dx = 1 + x + \frac{x^2}{2}$$

2nd approximation -

$$y_2 = 1 + \int_0^x f(x, y_1)\, dx$$

or $$y_2 = 1 + \int_0^x \left(x+1+x+\frac{x^2}{2}\right) dx = 1+x+x^2+\frac{x^3}{6}$$

3rd approximation -

$$y_3 = 1 + \int_0^x f(x, y_2)\, dx$$

or $$y_3 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

4th approximation -

$$y_4 = 1 + \int_0^x f(x, y_3)\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

5th approximation -

$$y_5 = 1 + \int_0^x f(x, y_4)\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

Example - Find the solution of the differential equation

$$\frac{dy}{dx} = x^2+y^2 \quad \text{for } x = 0.4 \text{ when } y(0) = 0$$

(correct upto four decimal position)

Sol Here, $f(x,y) = x^2+y^2$, $x_0=0$, $y_0=0$ & $x=0.4$

According to Picard's equation, the integral equation is

$$y_{m+1} = y_0 + \int_{x_0}^{x} f(x, y_m)\, dx$$

or

$$y_{m+1} = 0 + \int_{0}^{0.4} (x^2+y_m^2)\, dx$$

$$y_{m+1} = \int_{0}^{0.4} (x^2+y_m^2)\, dx$$

1st approximation -

$$y_1 = \int_{0}^{0.4} (x^2+y_0^2)\, dx$$

or

$$y_1 = \int_{0}^{0.4} (x^2+0^2)\, dx$$

$$y_1 = \left[\frac{x^3}{3}\right]_0^{0.4} = \frac{(0.4)^3}{3} = 0.0213$$

2nd Approximation -

$$y_2 = \int_{0}^{0.4} (x^2+y_1^2)\, dx$$

or

$$y_2 = \int_{0}^{0.4} (x^2+(0.0213)^2)\, dx$$

or
$$y_2 = \left[\frac{x^3}{3} + (0.0213)^2 x\right]_0^{0.4}$$

$$y_2 = \frac{(0.4)^3}{3} + (0.0213)^2 \times 0.4 = 0.0215$$

3rd Approximation –

$$y_3 = \int_0^{0.4} (x^2 + y_2^2)\, dx$$

or
$$y_3 = \int_0^{0.4} \left[x^2 + (0.0215)^2\right] dx$$

$$y_3 = \left[\frac{x^3}{3} + (0.0215)^2 x\right]_0^{0.4}$$

$$= \frac{(0.4)^3}{3} + (0.0215)^2 \times 0.4 = 0.0215$$

Hence, the value of $y(0.4) = 0.0215$ Ans

Runge Kutta Method :– It is introduced by the German mathematicians C. Runga and M.W. Kutta. It is also used for the approximation of solution of ordinary differential equations.

Runga Kutta method can be divided into four types –

1. First order R-K method
2. Second order R-K method
3. Third order R-K method
4. Fourth order R-K method

1) First Order Runga-Kutta Method :-

Let the differential Equation

$$\frac{dy}{dx} = f(x,y) \quad \text{with} \quad y(x_0) = y_0 \quad \text{—— (I)}$$

Euler's method gives

$$y_1 = y_0 + hf(x_0, y_0)$$

or $$y_1 = y_0 + hy_0' \quad \text{—— (II)} \quad (\because y' = f(x,y))$$

using Taylor's series, we get

$$y_1 = y(x_n + h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \dots \quad \text{—— (III)}$$

from eq^n (II) & (III), it is clear that Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, Euler's method is the Runge-Kutta method of the first order.

2) Second Order Runga-Kutta Method :-

Consider the differential Equation

$$\frac{dy}{dx} = y' = f(x,y) \quad \text{with} \quad y(x_0) = y_0 \quad \text{—— (I)}$$

Let h be the interval between equidistance value of x

In 2nd order R-K method, incremented y is computed from the formula -

$$\boxed{y_1 = y_0 + \frac{1}{2}(K_1+K_2)}$$

where,

$$K_1 = h \cdot f(x_0, y_0)$$

& $$K_2 = hf(x_0+h, y_0+K_1)$$

and $$\boxed{x_1 = x_0 + h}$$

Similarly, next incremented is calculated as-

$$\boxed{y_2 = y_1 + \frac{1}{2}(K_1+K_2)}$$

where $$K_1 = hf(x_1, y_1)$$

& $$K_2 = hf(x_1+h, y_1+K_1)$$

and $$\boxed{x_2 = x_1 + h}$$

and similarly for next interval.

3) <u>Third Order Runga Kutta Method</u> :-

In third order Runga-Kutta method, three factors are calculated as -

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

& $$K_3 = hf(x_0+h, y_0+K')$$

where $$K' = hf(x_0+h, y_0+K_1)$$

So, the third order Runga-Kutta formula is -

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)}$$

and similarly, we can find out next increment value of y.

4) <u>Fourth Order Runga-Kutta Method :-</u>

It is most commonly used form of Runga-Kutta method.

Let the differential equation of 1st order is -

$$\frac{dy}{dx} = y' = f(x, y) \quad \text{with initial condition } y(x_0) = y_0$$

Next value of y_1 is calculated as -

$$\boxed{y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)}$$

where,

$$K_1 = h \cdot f(x_0, y_0)$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

& $$K_4 = h \cdot f(x_0 + h, y_0 + K_3)$$

and $\boxed{x_1 = x_0 + h}$

Next increment of y is calculated same as above.

Example - Given the following differential equation $\frac{dy}{dx} = \frac{2-y^2}{5x}$, with $y(4) = 1$. Compute $y(4.1)$, $y(4.2)$ and $y(4.3)$ using Runga - Kutta second order method.

Solution - Here, $f(x,y) = \frac{2-y^2}{5x}$

$x_0 = 4$, $y_0 = 1$ and $h = 0.1$

1st approximation -

$$K_1 = h.f(x_0, y_0)$$
$$= 0.1.f(4,1)$$
$$= 0.1\left[\frac{2-1}{5\times4}\right] = 0.005$$

$$K_2 = h.f(x_0+h, y_0+K_1)$$
$$= 0.1 \cdot f(4.1, 1.005)$$
$$= 0.1\left[\frac{2-(1.005)^2}{5\times4.1}\right] = 0.004829$$

$$\therefore \quad y(4.1) = y_1 = y_0 + \frac{1}{2}(K_1+K_2)$$
$$= 1 + \frac{1}{2}(0.005 + 0.004829) = 1.004915$$

$$x_1 = x_0 + h = 4 + 0.1 = 4.1$$

2nd approximation :-

$$K_1 = h \cdot f(x_1, y_1)$$

$$= 0.1\, f(4.1, 1.004915)$$

$$= 0.1 \left[\frac{2 - (1.004915)^2}{5 \times 4.1} \right] = 0.004829$$

$$K_2 = h \cdot f(x_1 + h,\ y_1 + K_1)$$

$$= 0.1 \cdot f(4.2,\ 1.053214)$$

$$= 0.1 \left[\frac{2 - (1.053214)^2}{5 \times 4.2} \right] = 0.004241$$

$$\therefore \quad y(4.2) = y_2 = y_1 + \frac{1}{2}(K_1 + K_2)$$

$$= 1.004915 + \frac{1}{2}(0.004829 + 0.004241)$$

$$= 1.00945$$

$$x_2 = x_1 + h = 4.1 + 0.1 = 4.2$$

3rd approximation :-

$$K_1 = h \cdot f(x_2, y_2)$$

$$= 0.1 \cdot f(4.2, 1.00945)$$

$$= 0.1 \left[\frac{2 - (1.00945)^2}{5 \times 4.2} \right] = 0.004671$$

$$K_2 = h \cdot f(x_2 + h,\ y_2 + K_1)$$

$$= 0.1\, f(4.3, 1.01412)$$

$$K_2 = 0.1\left[\frac{2-(1.01412)^2}{5\times 4.3}\right] = 0.004518$$

$$\therefore \quad y(4.3) = y_3 = y_2 + \frac{1}{2}(K_1+K_2)$$

$$= 1.00945 + \frac{1}{2}(0.004671 + 0.004518)$$

$$= 1.01404$$

Hence,

$$\left.\begin{aligned} y(4.1) &= 1.004915 \\ y(4.2) &= 1.00945 \\ y(4.3) &= 1.01404 \end{aligned}\right\} \quad \underline{\underline{\text{Ans}}}$$

Example – Given $\frac{dy}{dx} = y-x$, $y(0)=2$. Find $y(0.1)$ and $y(0.2)$ correct to four decimal places. using R-K fourth order.

Sol – Here, $f(x,y) = y-x$,

$x_0 = 0$, $y_0 = 2$ and $h = 0.1$

1st approximation –

$$K_1 = h.f(x_0, y_0)$$

$$= 0.1 f(0,2)$$

$$= 0.1(2-0) = 0.2$$

$$K_2 = h.f\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$$
$$= 0.1\, f(0.05, 2.1)$$
$$= 0.1\,(2.1-0.05) = 0.205$$

$$K_3 = h.f\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$$
$$= 0.1\, f(0.05, 2.1025)$$
$$= 0.1\,[2.1025-0.05] = 0.20525$$

$$K_4 = h.f(x_0+h, y_0+K_3)$$
$$= 0.1\, f(0.1, 2.20525)$$
$$= 0.1\,[2.20525-0.1] = 0.210525$$

So,

$$y(0.1) = y_1 = y_0 + \frac{1}{6}(K_1+2K_2+2K_3+K_4)$$
$$= 2+\frac{1}{6}(0.2+2\times 0.205+2\times 0.20525+0.210525)$$
$$= 2+0.20517 = 2.20517 \approx 2.2052 \text{ Ans}$$

& $x_1 = x_0+h = 0+0.1 = 0.1$

2nd Approximation –

$$K_1 = h.f(x_1, y_1)$$
$$= 0.1\, f(0.1, 2.2052)$$
$$= 0.1\,[2.2052-0.1] = 0.21052$$

$$K_2 = h \cdot f\left(x_1+\frac{h}{2},\ y_1+\frac{K_1}{2}\right)$$

$$= 0.1\ f\,(0.15,\ 2.31046)$$

$$= 0.1\ [2.31046 - 0.15] = 0.2160$$

$$K_3 = h \cdot f\left(x_1+\frac{h}{2},\ y_1+\frac{K_2}{2}\right)$$

$$= 0.1\ f\,(0.15,\ 2.3132)$$

$$= 0.1\ [2.3132 - 0.15] = 0.2163$$

$$K_4 = h \cdot f\,(x_1+h,\ y_1+K_3)$$

$$= 0.1\ f\,(0.2,\ 2.4215)$$

$$= 0.1\ [2.4215 - 0.2] = 0.2221$$

Hence

$$y(0.2) = y_2 = y_1 + \frac{1}{6}(K_1+2K_2+2K_3+K_4)$$

$$= 2.2052 + \frac{1}{6}\left[0.2105 + 2\times 0.2160 + 2\times 0.2163 + 0.2221\right]$$

$$= 2.2052 + 0.2162$$

$$= 2.4214 \quad \text{Ans}$$