Graph Theory

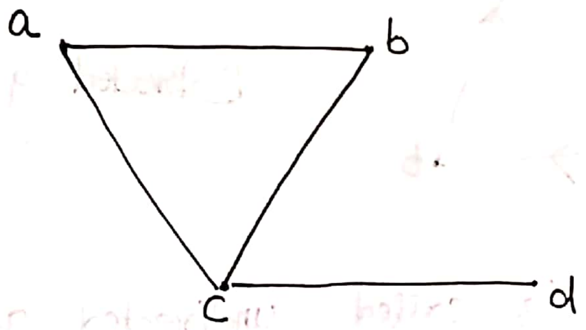
A Graph G consist of a non-empty set of vertices and set of edges.

i.e. $G = (V, E)$

where V = set of vertices or nodes.

and E = set of edges.

Example -



where $V = \{a, b, c, d\}$

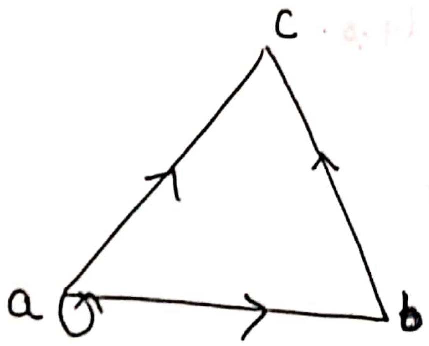
& $E = \{(a,b), (a,c), (b,c), (c,d)\}$

Two vertices are called adjacent if they are connected by an edge.

Two edges are called incident, if they share a common vertex.

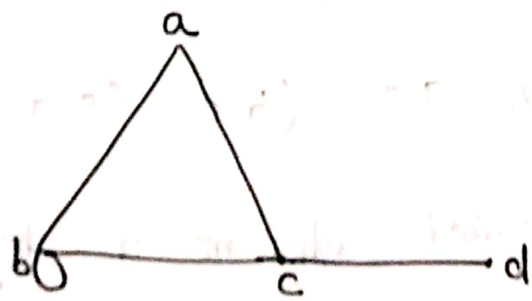
Directed and Undirected Graph :-

A graph $G = (V, E)$ is called a directed Graph or digraph if edge is associated with an ordered pair of vertices. That means each edge of the graph G has a direction which is represented by an arrow.



Directed graph

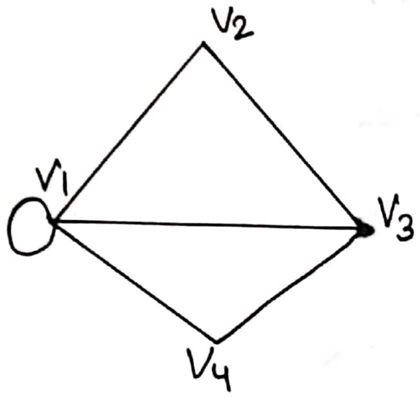
A graph G is called undirected graph if each edge is associated with an unordered pair of vertices.



undirected Graph

Degree :- The degree of a vertex of an undirected graph is the number of edges incident with it, except the self loop which contributes twice to the degree of the vertex. The degree of the vertex v in a graph G is denoted by $\deg_G(v)$

Example -



Here $\deg(v_1) = 5$
$\deg(v_2) = 2$
$\deg(v_3) = 3$
$\deg(v_4) = 2$

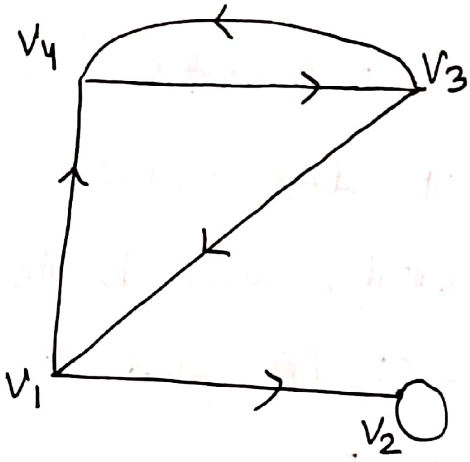
In case of directed graph, the degree is further classified as in-degree and out-degree.

The in-degree of a vertex is the number of edges ending at vertex v, and it is denoted by $\deg^-(v)$ and out-degree is the number of edges beginning at vertex v and it is expressed by $\deg^+(v)$.

Note → 1) self loops are counted 1 in-degree and 1 out-degree.

2) A vertex of degree 0 is called isolated vertex and a vertex is pendent if it has degree 1.

Example –



$deg^+(v_1) = 2$ $\quad$ $deg^-(v_1) = 1$

$deg^+(v_2) = 1$ $\quad$ $deg^-(v_2) = 2$

$deg^+(v_3) = 2$ $\quad$ $deg^-(v_3) = 1$

$deg^+(v_4) = 1$ $\quad$ $deg^-(v_4) = 2$

Hence,

sum of in-degree = sum of out degree

Handshaking Theorem :–

Handshaking theorem states that the sum of degrees of the vertices of a graph is twice the number of edges. If $G = (V, E)$ be a graph with e edges then

$$\sum deg_G(v) = 2e$$

i.e. sum of degree of the vertices in an undirected graph is even.

Proof – Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge get counted twice, once at each end.

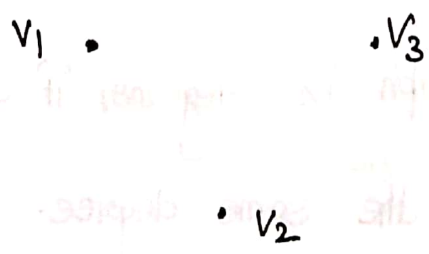
Thus the sum of the degree is equal twice the number of edges.

Note – In a simple graph, maximum number of edges with n vertices is – $\frac{n(n-1)}{2}$

Types of Graph :– There are different types of graphs which are as follows –

1) **Null Graph** – A graph which contain no edges is called null graph. The null graph of n vertices is denoted by N_n.

Example –

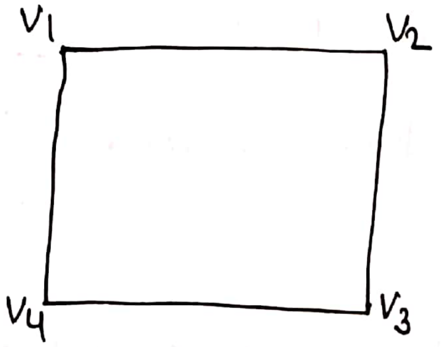


N_3 null graph

each vertex of null graph is isolated.

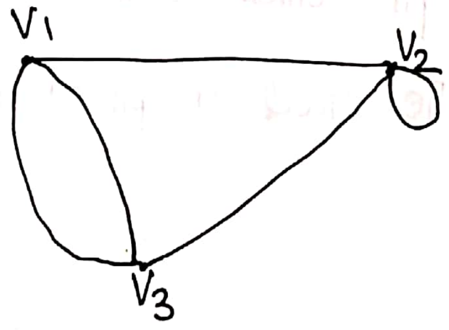
2) <u>Simple Graph</u> :- A graph is called simple (or strict) graph if graph is undirected vertex do not contain any self loops or multiple edges.

<u>Example</u>



3) <u>MultiGraph</u> :- A graph in which multiple edges may connect the same pair of vertices is called a multigraph. In other words, it is a graph having atleast one loop or multiple edges.

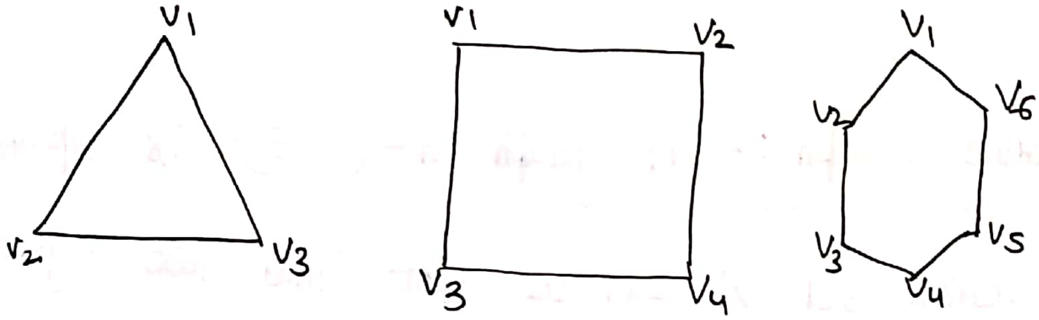
<u>Example</u>



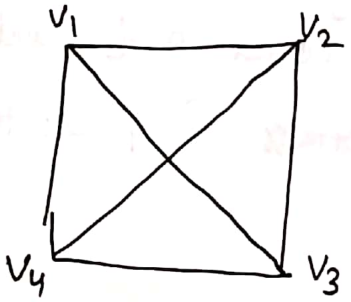
4) <u>Regular Graph</u> :- A graph is regular if all the vertices of the graph have the same degree.

If the degree of each vertex is r, then the graph is called a regular graph of degree r.

Example



regular graph of degree 2

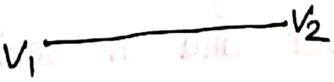


regular graph of degree 3

Note- Every null graph is regular of degree zero.

5) Complete Graph :- A graph G is called complete graph if every vertex of G is connected with every other vertex. A complete graph is usually denoted by K_n, where n is number of vertices.

K_1



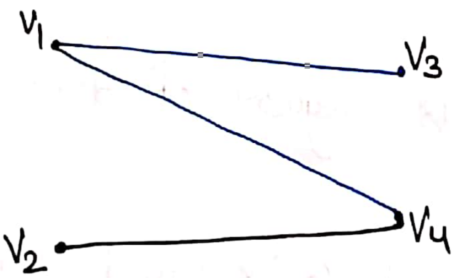
K_2

K_3

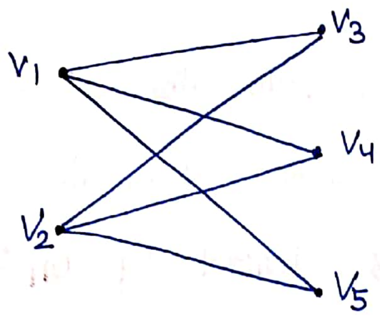
Number of edges of K_n complete graph is

$$\frac{n(n-1)}{2}.$$

6) <u>Bipartite Graph</u> :- A graph $G = (V, E)$ is bipartite if the vertex set V can be split into two disjoint sets, V_1 and V_2 in such a way that each edge in a graph joins a vertex in V_1 to a vertex in V_2 and no edges in G connects either two vertices of V_1 or two vertices of V_2.

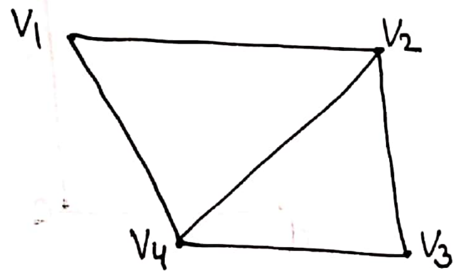


7) <u>Complete Bipartite Graph</u> :- A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to every single vertex in the second set. It is denoted by $K_{m,n}$ where m vertices in the first set and n vertices in the second set.

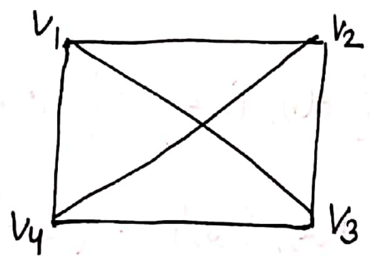


$K_{2,3}$

Planer Graph :- A graph G is called a planar graph if it can be drawn in a plane without edge crossing.



Non-Planar Graph :- A graph is non-planer if it cannot be drawn in a plane without graph edges crossing.



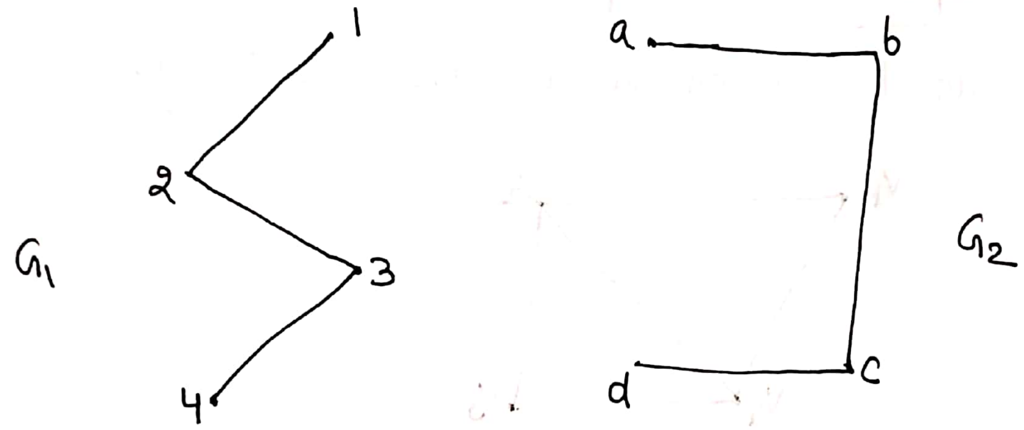
Isomorphism Graph :- Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a function $f: V_1 \rightarrow V_2$ such that

i) f is one-one, onto i.e. bijective.

ii) $\{a, b\}$ is an edge in E_1, iff $\{f(a), f(b)\}$ is an edge in E_2 for $a, b \in V_1$.

If two graphs G_1 and G_2 contain the same number of vertices connected in the same way, they are called isomorphic graphs. It is denoted by $G_1 \cong G_2$.

Example –



Here, $f(1) = a$, $f(2) = b$, $f(3) = c$ & $f(4) = d$

$\therefore$ f is one–one onto

further, $\{1,2\} \in E(G_1)$ & $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2,3\} \in E(G_1)$ & $\{f(2), f(3)\} = \{b, c\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ & $\{f(3), f(4)\} = \{c, d\} \in E(G_2)$

Hence, G_1 and G_2 are isomorphic.

i.e. $\boxed{G_1 \cong G_2}$

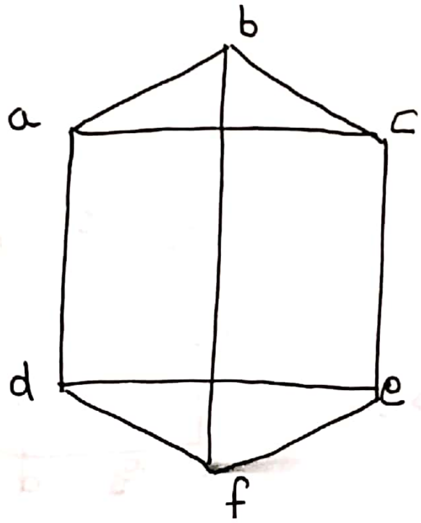
proved

Sub-Graph :- Consider a graph $G = (V, E)$

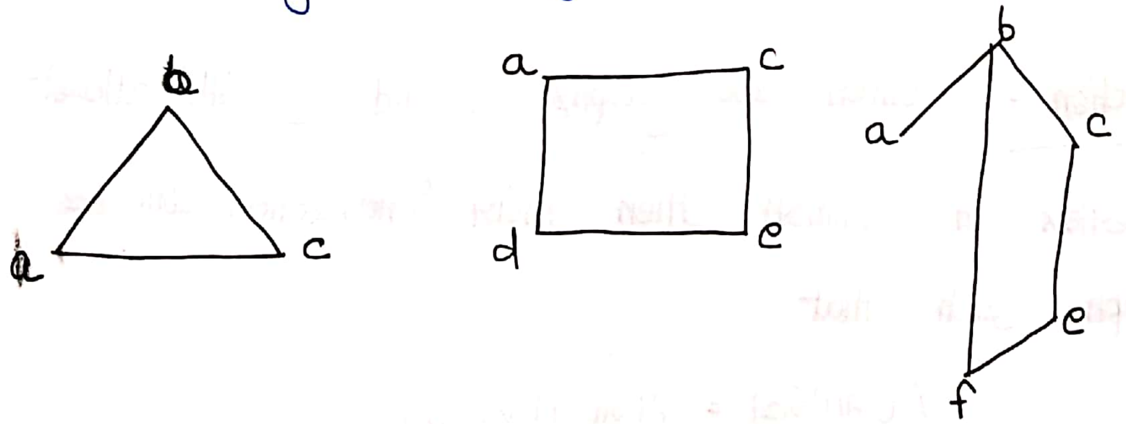
A graph $H = (V', E')$ is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G.

i.e. $V' \subseteq V$ and $E' \subseteq E$

Example Consider the graph G is –



– The following are subgroups of the above graph as –



Operations of Graphs — Some operations on the graph is as follows —

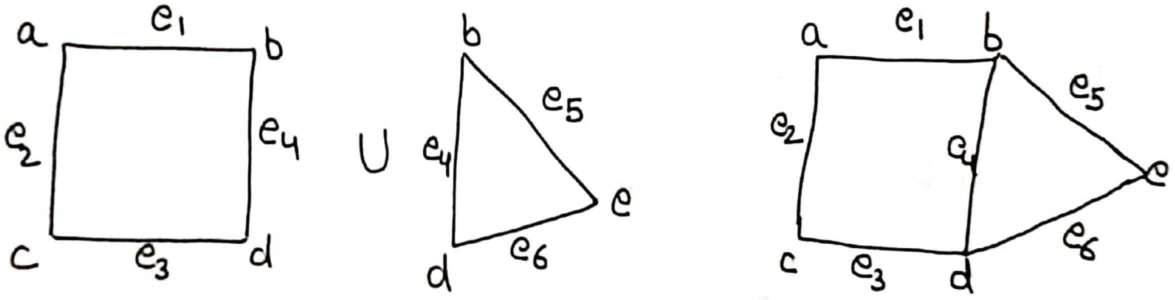
1) **Union** – Let two graphs G_1 and G_2 then union of G_1 & G_2 is denoted by $G_1 \cup G_2$ which is also a graph such that

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

&

$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$

Example –



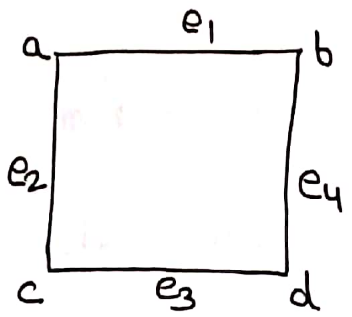
G_1 $\qquad$ G_2 $\qquad$ $G_1 \cup G_2$

2) **Intersection** – Given two graphs G_1 and G_2 with atleast one vertex in common then their intersection will be a graph such that

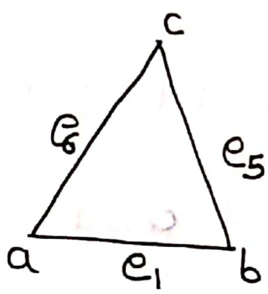
$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

&

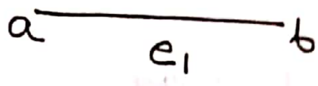
$$E(G_1 \cap G_2) = V(G_1) \cap E(G_2)$$



G_1



G_2

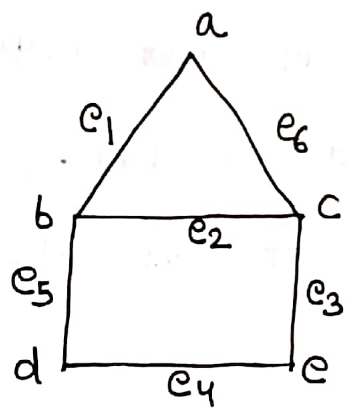


$G_1 \cap G_2$

Some terms of Graph :-

1) **walk** – A walk is defined as a finite length alternating sequence of vertices and edges. The total number of edges covered in a walk is called as length of the walk.

Example –



→ ~~$a e_1 b e_2 c e_3 e e_4 d$~~ length = 4

$a\ e_1\ b\ e_2\ c\ e_3\ e\ e_4\ d$

length = 4

or

$e\ e_3\ c\ e_2\ b\ e_1\ a\ e_6\ c\ e_3\ e\ e_4\ d$

length = 6

– A walk is called open walk if the starting and ending vertex are different.

– when walk starts and ends with same vertex then it is known as closed walk.

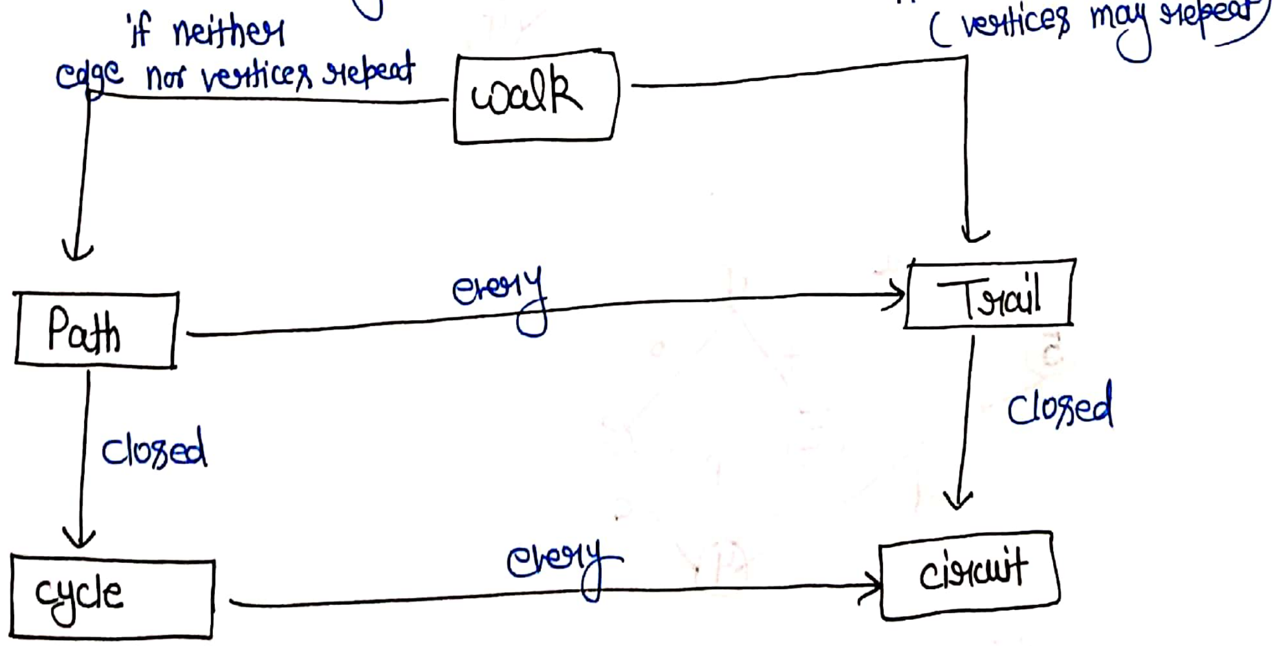
Path — A path is defined as an open walk in which neither vertices nor edges are allowed to repeat.

Cycle – A cycle is defined as a closed walk in which only repeated vertex is the first vertex and any other vertices and edges are not allowed to repeat.

i.e. a closed path is a cycle.

Circuit – A circuit is defined as the closed walk in which vertices may repeat but edges are not allowed to repeat.

i.e. Every cycle is a circuit but every circuit need not a cycle.



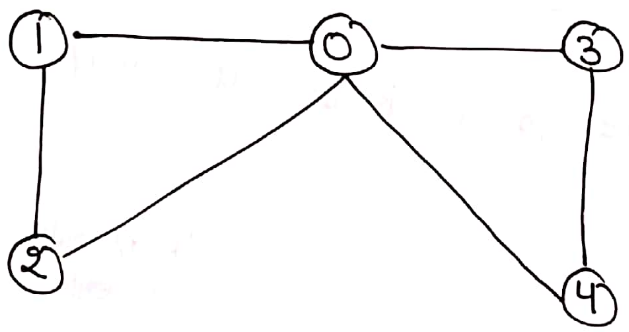
Eulerian Graph :- A graph which contains euler circuit is called eulerian graph.

Euler circuit – A trail which starts and ends at same vertex.

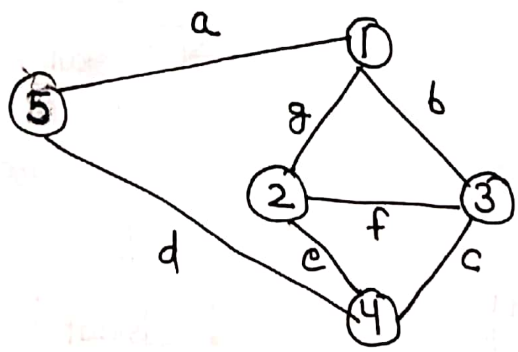
i.e. a walk which starts and ends at the same vertex and visit every edge at once.

(vertices may be repeated but edges not)

Example



euler graph



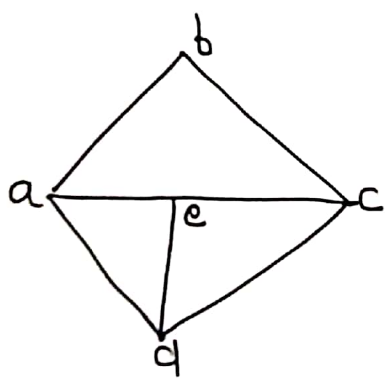
not euler graph (edges repeated)

Hamiltonian Graph – Hamiltonian graph is a graph which contains Hamiltonian circuit.

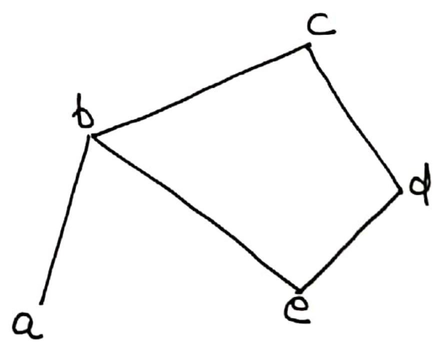
Hamiltonian Circuit – A simple circuit in G that passes through every vertex exactly once. (open & end with same vertex)

Hamiltonian Path – A simple path in G that passes through every vertex exactly once. (open & end at different vertex).

Example –



Hamiltonian circuit



Hamiltonian path

Note – 1) Every vertex in the graph has even degree. (condition for euler graph)

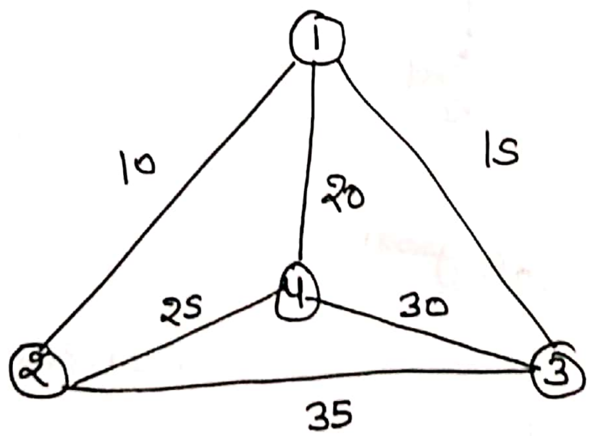
2) A complete graph with ~~two~~ more than two vertices is always Hamiltonian graph.

Travelling Salesman Problem :-

Travelling salesman problem (TSP) is an optimization problem in graph theory in which the nodes or vertex (cities) of a graph are connected by directed edges (routes), where the weight of an edge indicates the distance between two cities.

suppose a salesman area includes serveral cities and distance between every pair of cities, the problem is to find the shortest possible route that visits every city exactly once and returns to starting point. This problem is known as travelling salesman problem.

Example -



Travelling salesman tour in the Graph is

1 – 2 – 4 – 3 – 1

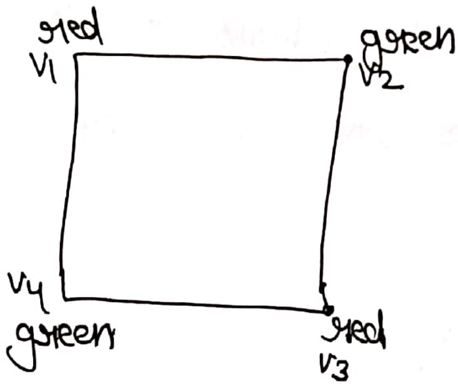
and cost of the tour is = 10 + 25 + 30 + 15 = 80

Chromatic Number – A minimum number of colors required for vertex coloring of graph G is called as the chromatic number of G and denoted by $X(G)$.

$X(G) = 1$ iff 'G' is a null graph otherwise $X(G) \geq 2$.

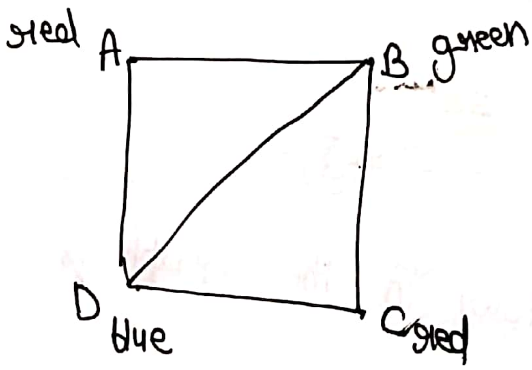
Vertex coloring is an assignment of colors to the vertices of a graph such that no two adjacent vertices have the same color.

Example –



$X(G) = 2$

or



$X(G) = 3$