Boolean Logic

Statement :- A statement or proposition is the basic building block of logic. It is defined as the declarative sentence that is either true or false but not both.

The truth value of a statement or proposition is True (denoted by T) if it is a true statement, and False (denoted by F) if it is a false statement.

Example - Consider the following statement -

1) The sun rises in the west.
2) $2+5=7$
3) All alphabets are vowel.

All the above sentences are propositions or statement where the first two are true and third one is False

Note – Questions, exclamations and commands are not propositions. For example –

(a) Close the door.

(b) what time is it ?

c) $x+1=2$.

The above sentences are not propositions.

To represent propositions, propositional variables are used. By convention, these variables are represented by small alphabets such as p, q, r, s ..... etc.

Compound Satement or Compound Proposition :–

A proposition obtained from the combinations of two or more propositions is called compound proposition.

i.e. compound proposition constructed using more than one atomic (or single) proposition.

Example – If it is raining then the game is cancelled.

Compound proposition is also called molecular proposition or composite proposition.

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Connectives — The words or symbols used to form compound proposition are called connectives. i.e. The propositions are combined together using logic connectives or logical operators.

There are five basic connectives which are as follows –

1) Negation
2) Conjunction
3) Disjunction
4) Conditional
5) Biconditional

Truth Tables :– A truth table is a table that shows the truth table (which is either true or false) of a compound proposition for all possible cases.

True values is denoted by T and false value is expressed by F.

Now, we will discuss all the connectives one by one —

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1) **Negation** :- If p is a proposition, then the negation of p is denoted by $\sim p$ and read as not p or negation of p.

when p is true then $\sim p$ is false and if p is false then $\sim p$ is true.

The truth table of $\sim p$ is -

p	$\sim p$
T	F
F	T

Example - Statement is —

p : It is raining today.

then negation is —

$\sim p$: It is not raining today.

Note - 1) Negation is not a connective, since it does not join two statements. That means, $\sim p$ is not really a compound statement.

2) There are some propositions that all have the same meaning which are as —

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p : All people are intelligent.

or p : Every person is intelligent.

or p : Each person is intelligent.

The negation is of, above proposition is –

~p : Some students are not intelligent.

or ~p : There exists a Student who is not intelligent.

or ~p : At least one Student is not intelligent.

3) One more example is as –

q : No student is intelligent.

~q : Some students are intelligent.

2) Conjunction :– For any two propositions p and q, their conjunction is denoted by $p \wedge q$ which means 'p and q'. The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false.

Truth table –

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

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Example - (a) p : It is cold.

q : It is raining.

then $p \wedge q$: It is cold and raining.

(b) p : $5x+6=26$.

q : $x>3$.

then $p \wedge q$: $5x+6=26$ and $x>3$.

3) Disjunction :– If p and q are two statements, then disjunction of p and q is denoted by '$p \vee q$' and read as 'p or q'. The disjunction $p \vee q$ is true when either p or q is true, otherwise false.

The truth table of $p \vee q$ is –

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Example - (a) p : It is cold.

q : It is raining.

then $p \vee q$: It is cold or raining.

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4) **Conditional or Implication** :- If p and q are two proposition, then the compound proposition 'if p and q' denoted by $p \Rightarrow q$ is called conditional proposition or implication and the connective is the Connective connective.

In $p \Rightarrow q$, proposition p is called hypothesis and proposition q is called Conclusion.

- The implication $p \Rightarrow q$ is false when p is true and q is false otherwise it is true.

Truth Table -

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Example - 1) p : The earth is round.

q : The earth travels the sun.

then $p \Rightarrow q$: If the earth is round then the earth travels the Sun.

2) p : It is raining.

q : The game is cancelled.

then $p \Rightarrow q$: If it is raining then the game is cancelled.

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5) **Biconditional or Double Implication** :- Let p and q are two propositions, the statement 'p if and only if (iff) q' is called a biconditional and it is denoted by $p \leftrightarrow q$. and the connective if and only if (iff) is the biconditional connective.

The implication $p \leftrightarrow q$ is true when proposition p and q have same value (either both are true or false) otherwise it is False.

The truth table of $p \leftrightarrow q$ is

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Example - p : He swims.
q : The water is warm.

then $p \leftrightarrow q$: He swims if and only if the water is warm.

Example - Construct truth table for

(i) $p \vee \sim q \Rightarrow p$

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Solution - (i) The truth table of the given compound statement is as —

p	q	$\sim q$	$p \vee \sim q$	$p \vee \sim q \Rightarrow p$
F	F	T	T	F
F	T	F	F	T
T	F	T	T	T
T	T	F	T	T

(ii) construction of another compound statement is —

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Example - Show that, $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$ using truth table.

Sol - These two expression are logically equivalent, if the two columns corresponding to the given two expression have identical truth values.

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Now, we construct truth table —

p	q	$p \Leftrightarrow q$	$p \vee q$	$p \wedge q$	$(p \vee q) \Rightarrow (p \wedge q)$
F	F	T	F	F	T
F	T	F	T	F	F
T	F	F	T	F	F
T	T	T	T	T	T

According to truth table

$$p \Leftrightarrow q \approx (p \vee q) \Rightarrow (p \wedge q)$$ proved

Example- Show that $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Sol - To prove the above equivalence, we prepare the following table —

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
F	F	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F

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Example – Construct truth table for

$$[\sim(p \wedge q) \vee r] \Rightarrow \sim p$$

Sol The truth table of the given compound statement is shown below –

p	$\sim p$	q	r	$p \wedge q$	$\sim(p \wedge q)$	$\sim(p \wedge q) \vee r$	$\sim(p \wedge q) \vee r \Rightarrow \sim p$
F	T	F	F	F	T	T	T
F	T	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	T	T	T	F	T	T	T
T	F	F	F	F	T	T	F
T	F	F	T	F	T	T	F
T	F	T	F	T	F	F	T
T	F	T	T	T	F	T	F

Converse, Inverse and Contrapositive :– Let p and q are two propositions then converse, inverse and contrapositive of the implication $p \Rightarrow q$ is –

(a) **Converse** — $q \Rightarrow p$

(b) **Inverse** — $\sim p \Rightarrow \sim q$

(b) **Contrapositive** — $\sim q \Rightarrow \sim p$

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The truth table of all the compound propositions as–

p	q	Conditional $p \Rightarrow q$	Converse $q \Rightarrow p$	Inverse $\sim p \Rightarrow \sim q$	contrapositive $\sim q \Rightarrow \sim p$
F	F	T	T	T	T
F	T	T	F	F	T
T	F	F	T	T	F
T	T	T	T	T	T

Tautologies and Contradictions :-

A compound propositions that is always true for all posible truth values (either true or false) of its variables then the propositions is called tautology and if it is always false for all possible truth values then it is known as contradictions.

Finally, a proposition that is neither a tautology nor a contradiction is called a contingency.

Example– Show that proposition $p \Rightarrow (p \vee q)$ is tautology.

Sol– The truth table of the given proposition is as —

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p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

since, from the truth table it is clear that all the values of given expression is true.

So, given proposition is tautology. proved

Example - Verify that the proposition $p \wedge (q \wedge \sim p)$ is a Contradiction.

Sol we construct the truth table of the given proposition —

p	q	$\sim p$	$q \wedge \sim p$	$p \wedge (q \wedge \sim p)$
F	F	T	F	F
F	T	T	T	F
T	F	F	F	F
T	T	F	F	F

since, All the values of last column are false.

So, the given proposition is contradiction.

Example - Use truth table, show that proposition $\sim p \vee q$ is contingency.

Sol

p	q	$\sim p$	$\sim p \vee q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

since, last column of truth table contains both true and false value.

So, $\sim p \vee q$ is contingency. proved

Ques - 1. Prove that the following propositions are tautologies -

(a) $p \Rightarrow (q \Rightarrow p)$

(b) $[p \wedge (p \Leftrightarrow q)] \Rightarrow q$

(c) $p \wedge (p \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(d) $(\sim p \Rightarrow \sim q) \Rightarrow (q \Rightarrow p)$

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Well Formed Formula (wff) :– A well formed formula or simply wff of propositional logic are obtained by using the following rules –

1) All variables and constants are well-formed formula.
2) If p is a well formed formula then $\sim p$ is also a well-formed formula.
3) If P and Q are well formed formula then $(P \vee Q)$, $(P \wedge Q)$, $(P \Rightarrow Q)$ and $(P \Leftrightarrow Q)$ are all well-formed formula.
4) A statement formula which can be obtained by finitely applying the above rules are well formed formula.

Example – $\sim (P \vee q)$ and $(P \Rightarrow (P \wedge q))$ are well formed formulae. where as –

$P \vee q$ and $(P \wedge q) \Rightarrow (\sim p)$ are not well formed formulae because parentheses are not used.

Quantifiers :– Quantifiers are words that refer to quantities such as some, few, many, all, none and indicate how frequently a certain statement is true.

There are two types of quantifier in mathematical logic —

(a) universal Quantifier (b) Existential Quantifier

1) **Universal Quantifier :-** universal quantifier states that the statements within its scope are true for every value of the specific variable.

It is denoted by the symbol $\forall$ and read as 'for all' or 'for each'.

Example - Consider the sentence -

'All human beings are mortal.'

Let $P(x)$ denote 'x is mortal'

then the above sentence can be written as

$$\forall x \, P(x)$$

2) **Existential Quantifier :-** Existential quantifier states that the statements within its scope are true for some values of the specific variable.

It is denoted by the symbol $\exists$ and read as 'there exists'

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Example - Consider the statement –

'Some people are dishonest.'

Let $P(x)$ denote 'People are dishonest.'

then the proposition expressed as –

$$\exists x \ P(x)$$

Normal Forms :– By comparing truth tables, we can determine whether two logical expression P and Q are equivalent. But the process is very lengthy when the number of variable increases.

So, we consider an alternate procedure known as the reduction to normal forms. The standard forms are called normal forms or canonical forms.

Types of Normal forms :– There are two types of normal form –

1) Disjunctive Normal Form
2) Conjunctive Normal Form

For Convenient we use –

product in place of Conjunction and
sum in place of disjunction

(SOP)

1) Disjunctive Normal Form (DNF) :-

In an logical expression, a product of the variables and their negations is called an elementary product. like $p \wedge \sim q$, $\sim p \wedge \sim q$, $p \wedge q$ etc.

A sum of the variables and their negations is called an elementary sum. i.e. $p \vee q$, $\sim p \vee q$ etc.

A logical expresion is said to be in disjunctive Normal Form (DNF) if it is the sum of elementary products.

Example – $p \vee (q \wedge r)$, $(p \wedge q) \vee (\sim q \wedge r)$

Some basic Laws of boolean expression –

1) Commutative Law – (a) $p \vee q \equiv q \vee p$

(b) $p \wedge q \equiv q \wedge p$

2) Associative Law – (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

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3) **Distributive Law** – (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

4) **Idempotent Law** - (a) $p \vee p \equiv p$

(b) $p \wedge p \equiv p$

5) **Involution Law** - $\sim(\sim p) \equiv p$

6) **De Morgan's Law** - (a) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

(b) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

7) **Complement Law** - (a) $p \vee \sim p \equiv$ True

(b) $p \wedge \sim p \equiv$ False

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2) **Conjunctive Normal Form (CNF)** (POS) :- A logic expression is said to be in Conjunctive Normal Form (CNF) if it is the product of elementary sums.

Example - $(p \vee q) \wedge (\sim p \vee r)$, $\sim p \wedge (p \vee q)$

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Procedure to obtain a DNF or CNF of a given Expression — Three steps are required which are as—

1) Remove all $\Rightarrow$ and $\Leftrightarrow$ by an equivalent expression containing the connectives $\wedge, \vee, \sim$ only.

2) Eliminate $\sim$ before sums and products by using the double negation or by using Demorgan's laws.

3) Apply the distributive law (if necessary)

Note — (a) $p \Rightarrow q \equiv \sim p \vee q$

(b) $p \Leftrightarrow q \equiv [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

Example — Obtain DNF of the followings —

(a) $p \wedge (p \Rightarrow q)$

(b) $p \Rightarrow ((p \Rightarrow q) \wedge \sim(\sim q \vee \sim p))$

Sol (a)

$$p \wedge (p \Rightarrow q)$$
$$\equiv p \wedge (\sim p \vee q)$$
$$\equiv (p \wedge \sim p) \vee (p \wedge q)$$ which is DNF

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(b) $p \Rightarrow ((p \Rightarrow q) \wedge \sim(\sim q \vee \sim p))$

$\equiv \sim p \vee ((\sim p \vee q) \wedge \sim(\sim q \vee \sim p))$

$\equiv \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$

$\equiv \sim p \vee [(\sim p \wedge (q \wedge p)) \vee q \wedge (q \wedge p)]$

$\equiv \sim p \vee [(\sim p \wedge q) \wedge p] \vee [(q \wedge q) \wedge p]$

$\equiv \sim p \vee F \vee (p \wedge q)$

$\equiv \sim p \vee (p \wedge q)$ DNF

Example obtain CNF of $p \wedge (p \Rightarrow q)$

Sol $p \wedge (p \Rightarrow q) \equiv p \wedge (\sim p \vee q)$

which is CNF

Principal DNF :- A DNF is called principal DNF if each term of DNF contain all the variables either original form or negation form.

In principal DNF each term is known as minterms. The number of minterms in n variables is 2^n.

Example - $(p \wedge q) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

The process for obtaining principal disjunctive form is –

1) <u>By truth table</u> –

(a) construct a truth table of a given compound propositions.

(b) For every truth value T of the given proposition, select the minterms.

(c) write all the minterms, by using disjunction or $\vee$.

2) <u>without using truth table</u> -

(a) obtain a disjunctive canonical form.

(b) Drop elementary products which are contradictions (such as $P \wedge \sim p$)

(c) If p_i or $\sim p_i$ are missing in an elementary product α, then replace α by $(\alpha \wedge p_i) \vee (\alpha \wedge \sim p_i)$.

d) Repeat step (c) until all elementary products are reduced to sum of minterms.

(e) Identical minterms appearing in the disjunction are deleted.

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Example – obtain principal DNF of $P \Rightarrow q$

Sol Construct truth table (method 1)

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

The column containing $P \Rightarrow q$ has T for three combination. So, principal DNF is –

$$(P \wedge q) \vee (\sim P \wedge q) \vee (\sim P \wedge \sim q)$$

without using truth table (method 2)

$$P \Rightarrow q$$

$$\equiv \sim P \vee q$$

$$\equiv (\sim P \wedge q) \vee (\sim P \wedge \sim q) \vee (P \wedge q) \vee (\sim P \wedge q)$$

$$\equiv (\sim P \wedge q) \vee (\sim P \wedge \sim q) \vee (P \vee q) \quad \text{Ans}$$

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Principal CNF – A CNF is called principal CNF if each term of conjunctive normal form contain each variable or its negation.

Each term is called maxterm.

Example – $(p \vee q) \wedge (\sim p \vee \sim q)$

Note – The process for obtaining principal CNF is similar to the one followed for principal DNF. only difference is that we consider false value of truth table and T denoted negative of variable & F express variables as it is.

Example – Obtain principal CNF for $p \wedge q$

Sol **Truth table** -

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

taking False values

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So, principal CNF is –

$(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$ Ans

without using truth table.

$p \wedge q$

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (p \vee q) \wedge (\sim p \vee q)$

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$ Ans

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